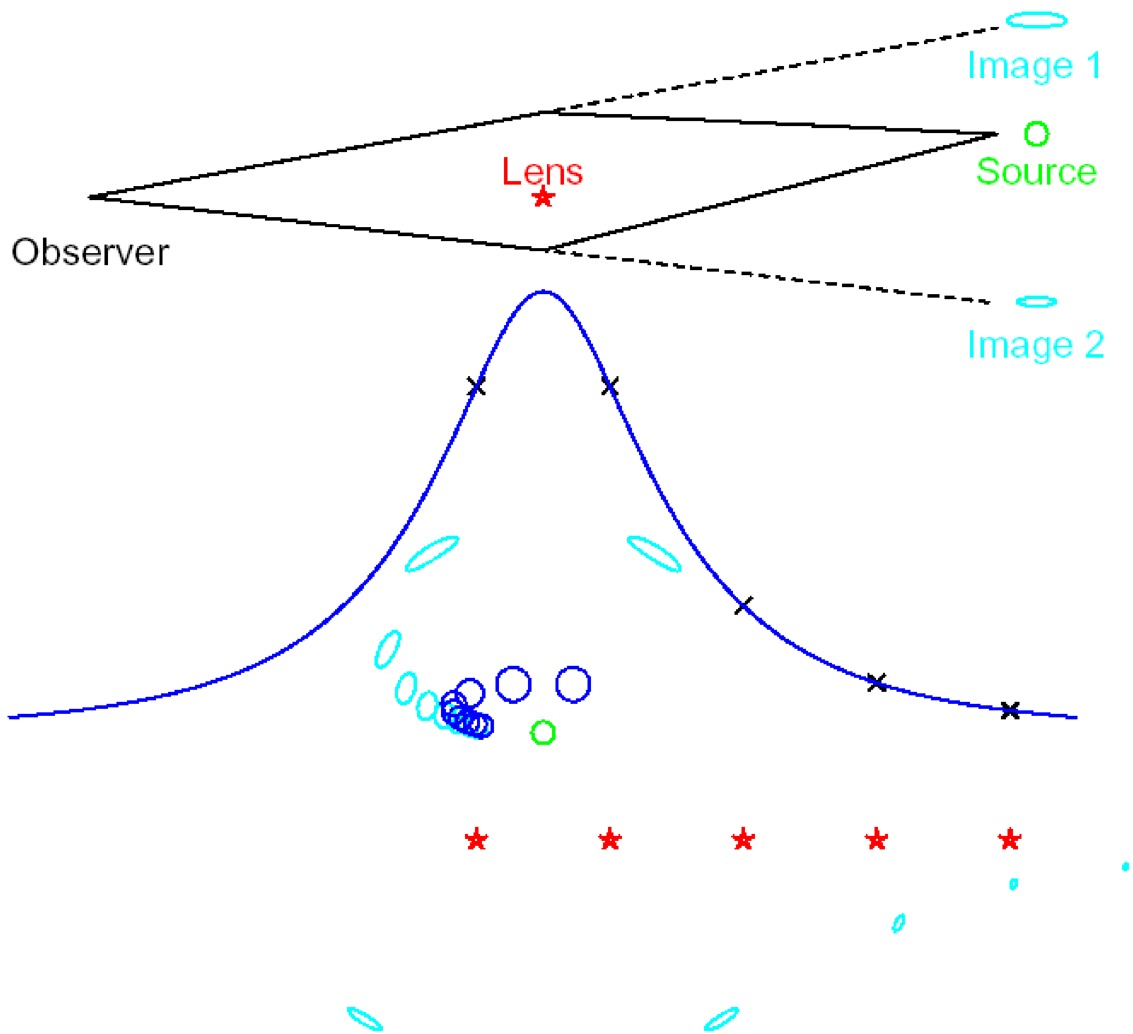
Microlensing II:
Planetary Microlensing Basics
Andy Gould (Ohio State)



Binary-Lens Equation

$$\mathbf{u} - \mathbf{y} = -\frac{\mathbf{y} - \mathbf{y}_L}{|\mathbf{y} - \mathbf{y}_L|^2}$$

$$\mathbf{y}_L = 0 \rightarrow \mathbf{u} - \mathbf{y} = -\frac{\mathbf{y}}{y^2} \implies u - y = -\frac{1}{y}$$

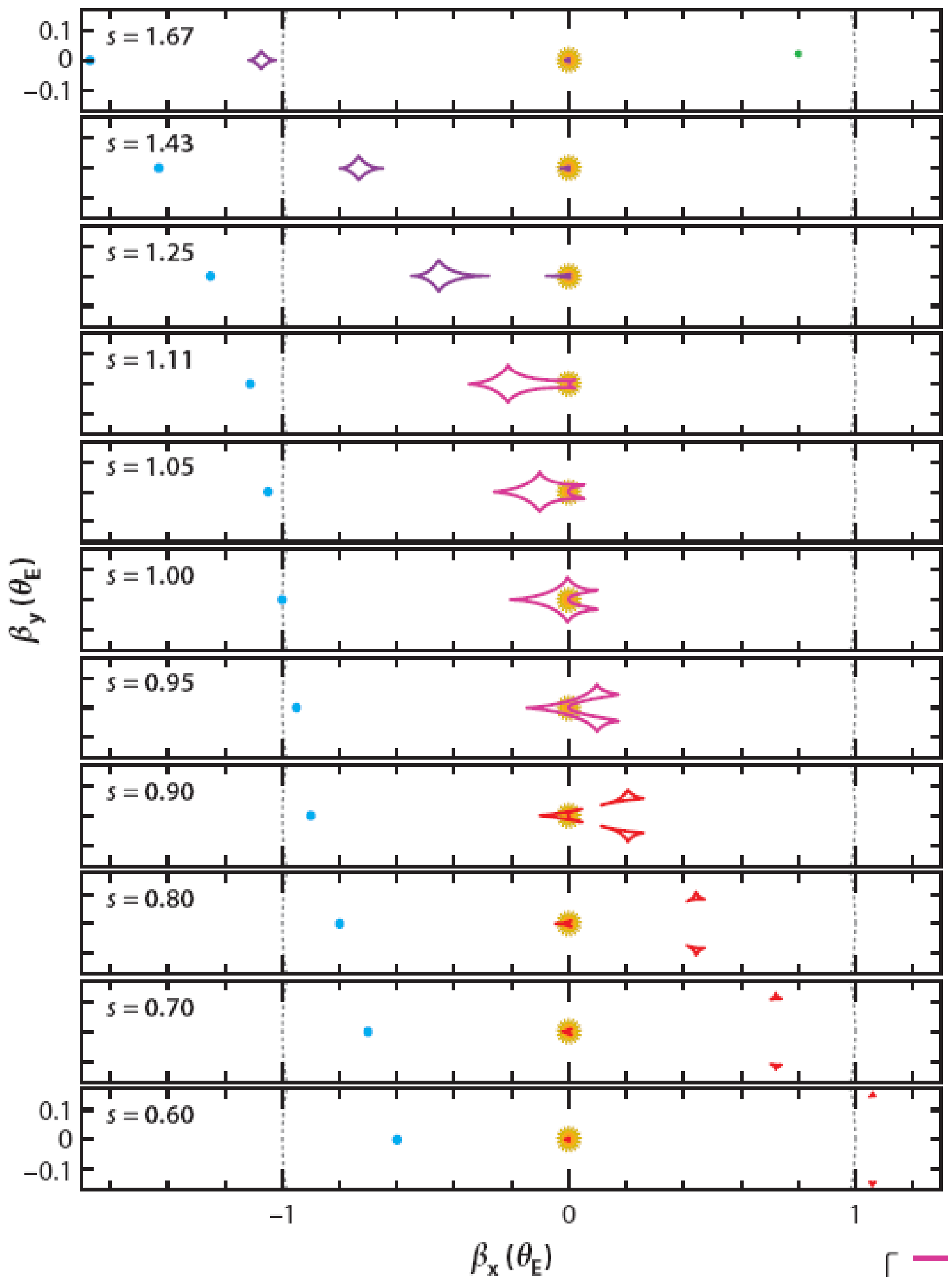
$$\implies (y - u)y = 1 \implies (\theta_I - \theta_S)\theta_I = \theta_{\mathrm{E}}^2$$

$$\mathbf{u} = \mathbf{y} - \sum_i \epsilon_i \frac{\mathbf{y} - \mathbf{y}_{m,i}}{|\mathbf{y} - \mathbf{y}_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{\mathrm{tot}}}$$

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + iu_2 \qquad z \equiv y_1 + iy_2$$

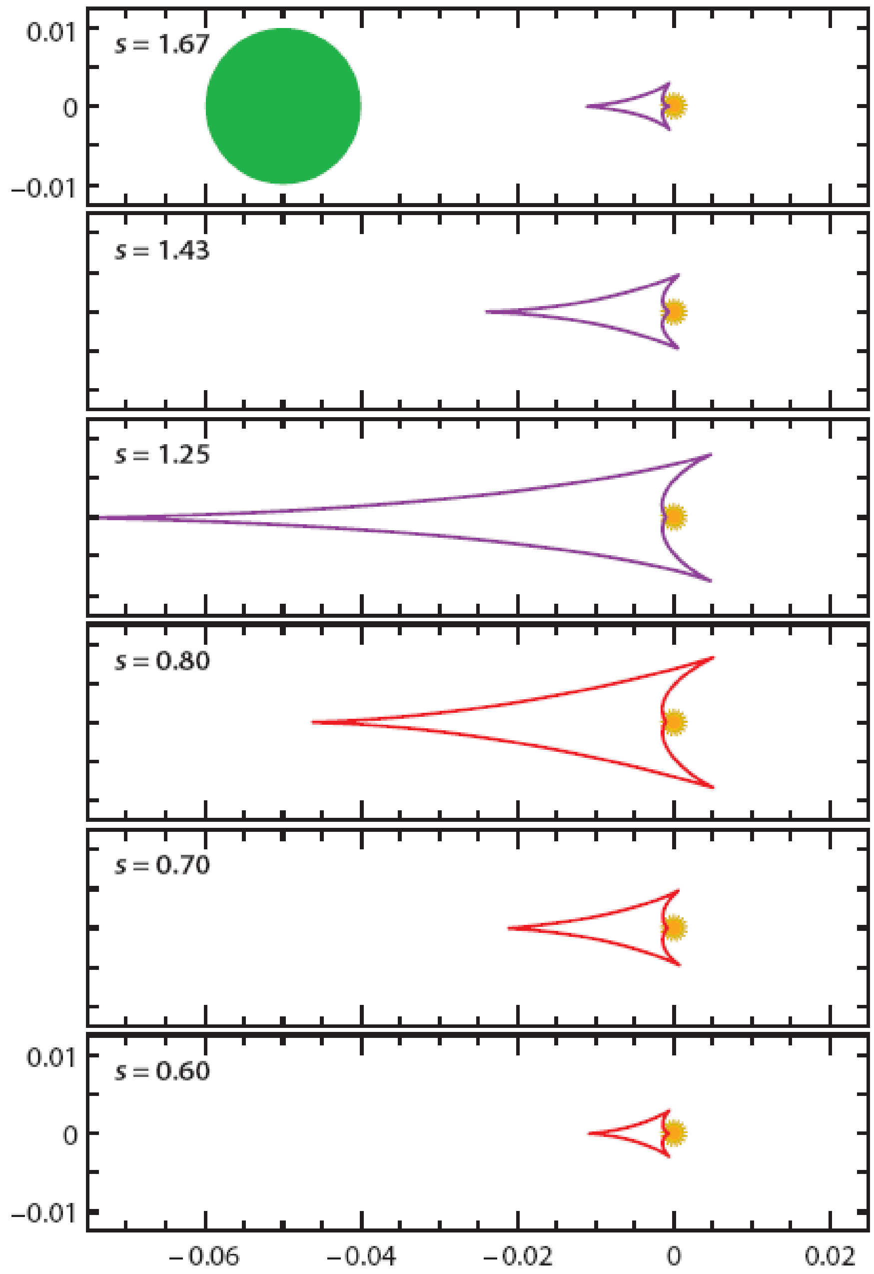
Binary-Lens Topologies



6 Features & 6 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q

Planetary Central Caustics



7 Features & 7 Parameters

Time of Peak	t_0
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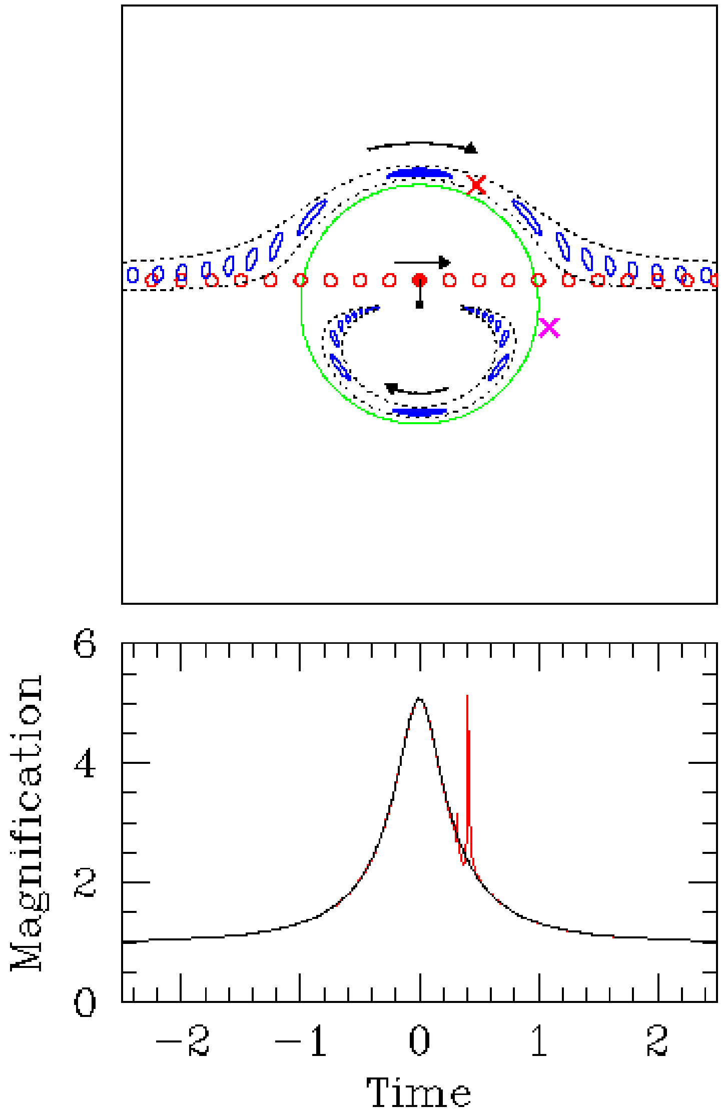
Smearing of Caustic	$t_* = \rho * t_E$

How Microlensing Finds Planets



Mao & Paczynski
Central Caustics

GRAVITATIONAL MICROLENSING BY DOUBLE STARS AND PLANETARY SYSTEMS

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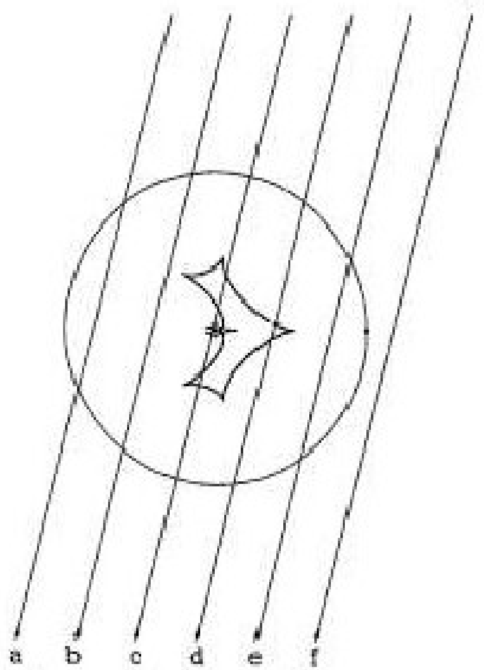
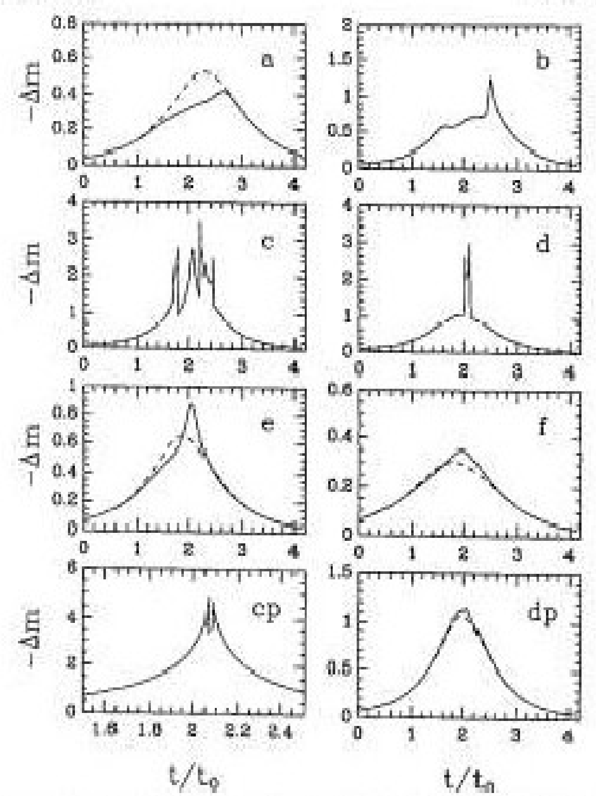
Received 1991 March 12; accepted 1991 April 2





. 1.—Geometry of microlensing by a binary, as seen in the sky. The
y star of 1 $M_\odot$ is located at the center of the figure, and the secondary of
or 0.001 $M_\odot$ is located on the right, on the Einstein ring of the
y. The radius of the ring is 1.0 mas for a source located at a distance of 8
d the lens at 4 kpc. The two complicated shapes around the primary are

the lens. The effect is strong even if the companion is a planet. A massive search for microlensing of the Galactic bulge stars may lead to a discovery of the first extrasolar planetary systems.

Gould & Loeb
Planetary Caustics

DISCOVERING PLANETARY SYSTEMS THROUGH GRAVITATIONAL MICROLENSES

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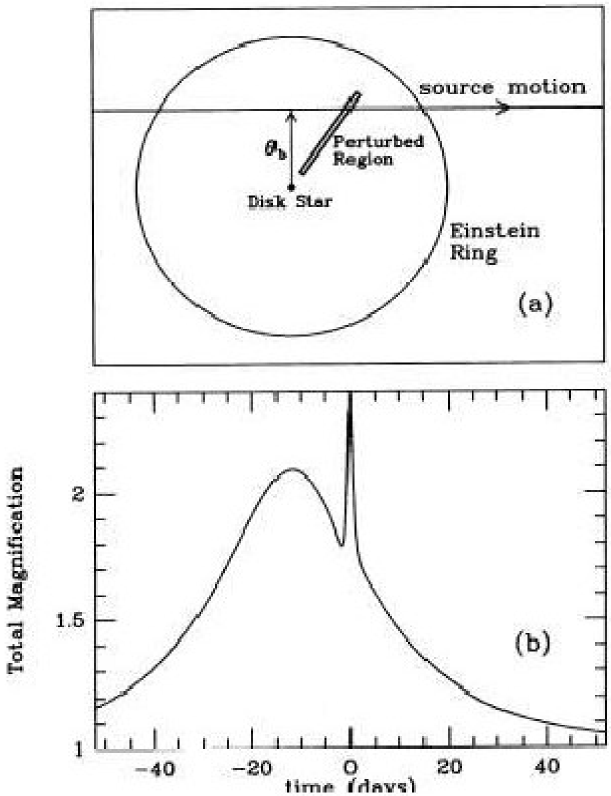


5. OBSERVATIONAL REQUIREMENTS

Two distinct steps are required to observe a planetary system by microlensing. First, one must single out a disk star which happens to be microlensing a bulge star. Second, one must observe this star often enough to catch the deviation in the light curve due to the planet. The first step involves the observation of millions of bulge stars on the order of once per day. The second step involves the observation of a handful of stars many times per day. In the following we give a rough outline of what is required for each of these steps.

While observations from one site would be useful, there are advantages to be gained by observing from several sites. First,

two telescopes that were totally committed. Third, in view of the fleeting nature of the events, it would seem prudent to build in some redundancy in case of bad weather at a particular site. Thus, the optimal scheme would employ, say, a dozen telescopes. Each of these would be committed to carry out two observations per night. During the near-December season,

6 Features & 6 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q

Clear Division Between

Known Knowns & Known Unknowns

Planet/star mass ratio

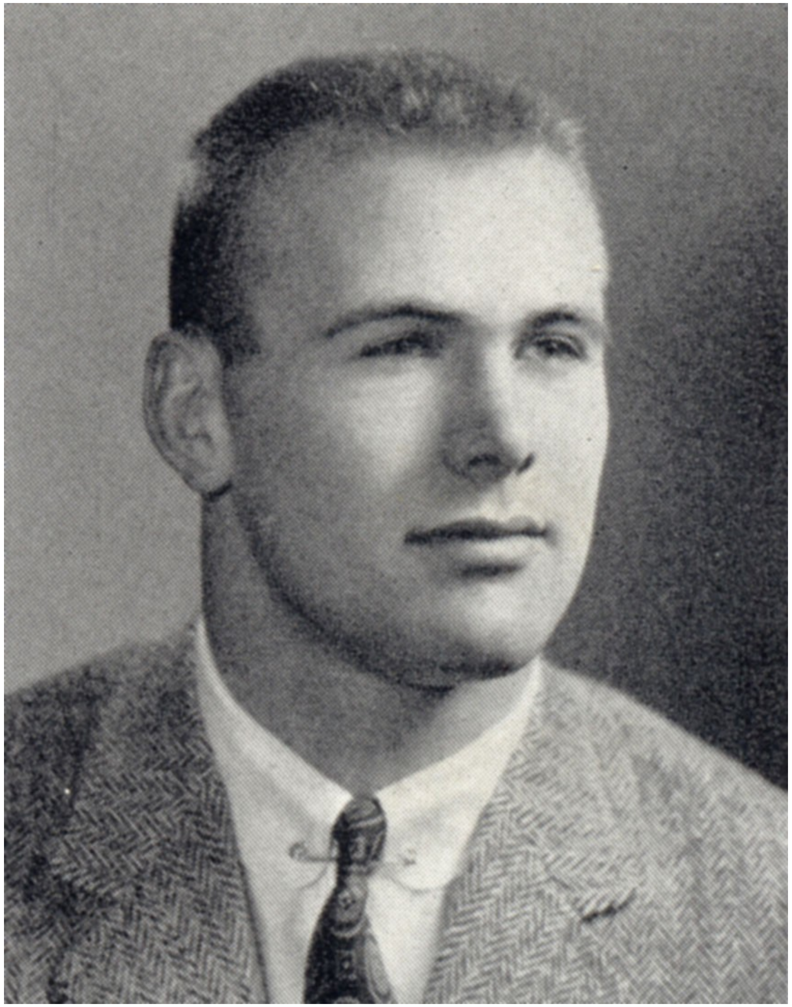
Separation in units of Einstein radius

Planet mass

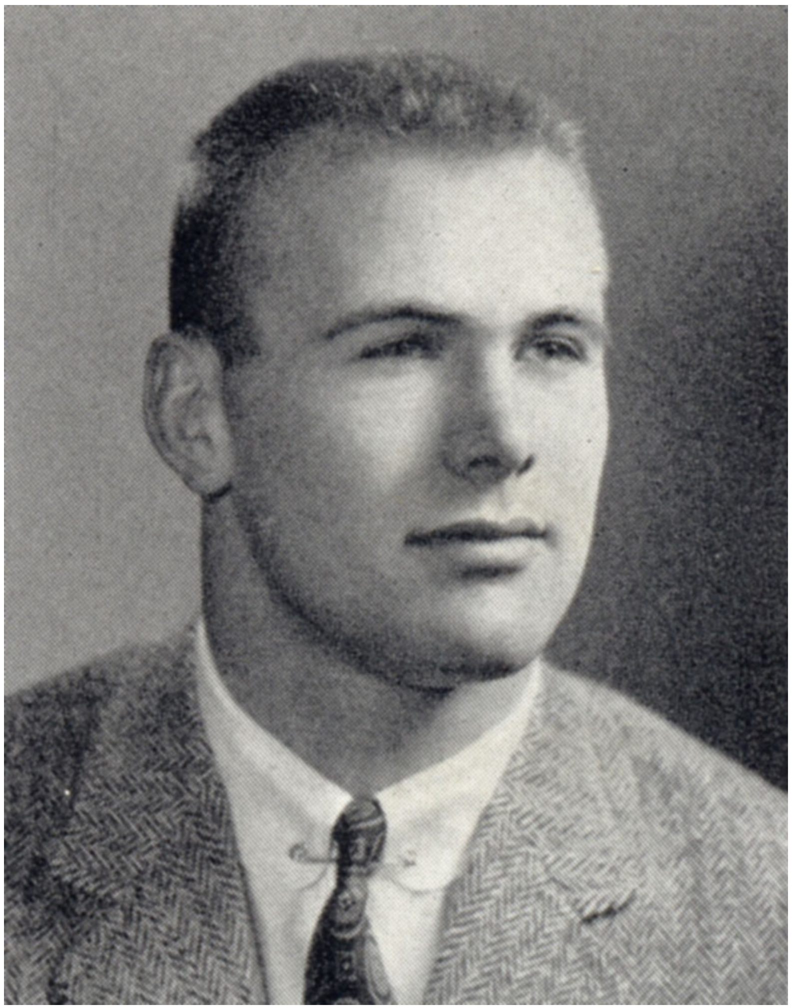
System distance

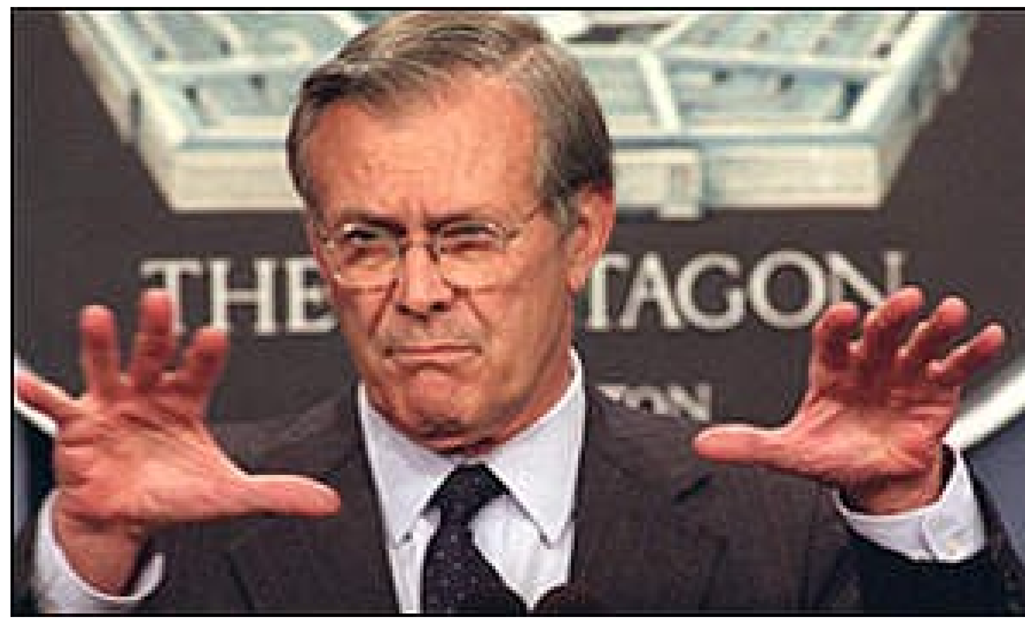
Planet/star projected physical separation

And inevitably



And inevitably: Unknown Unknowns





AP

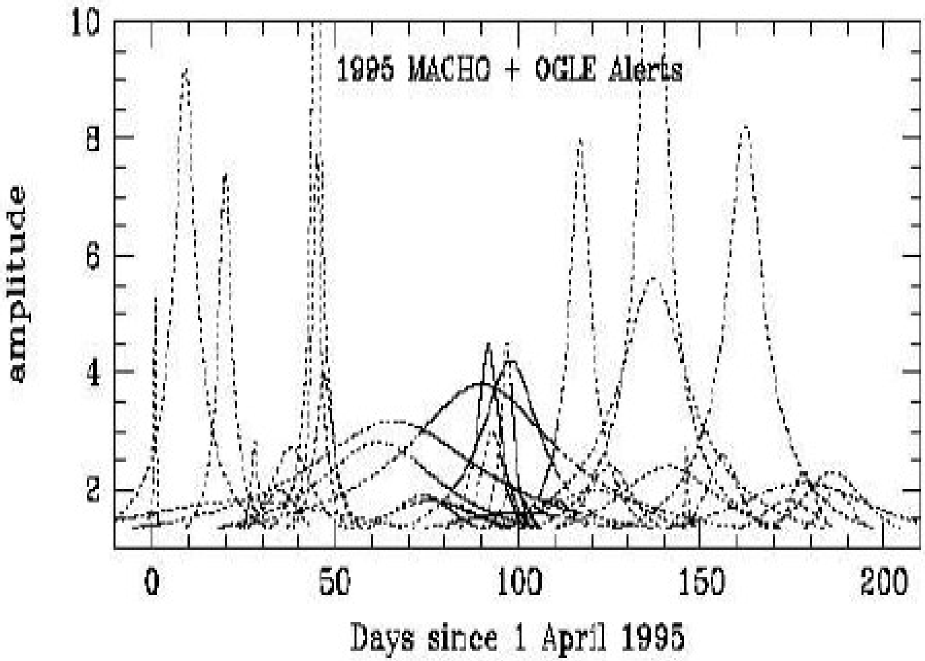
And inevitably: Unknown Unknowns

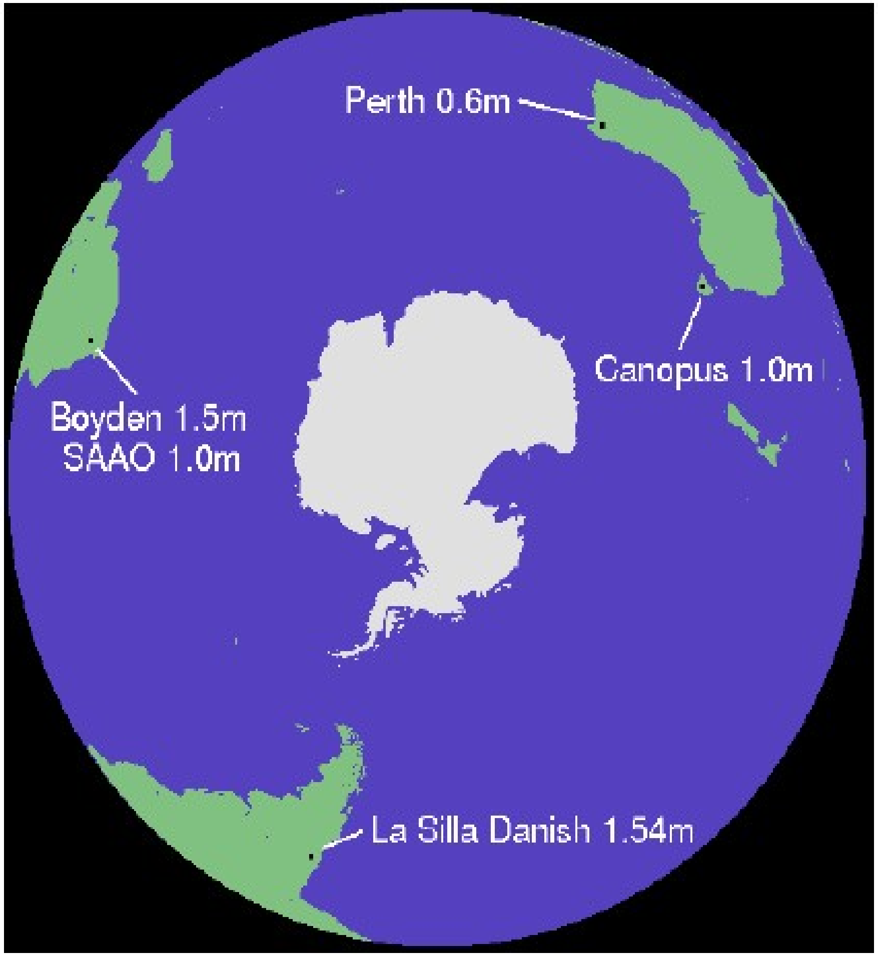
Planet/star 3-D separation

Planet Orbital Motion

1995 PLANET Pilot Season

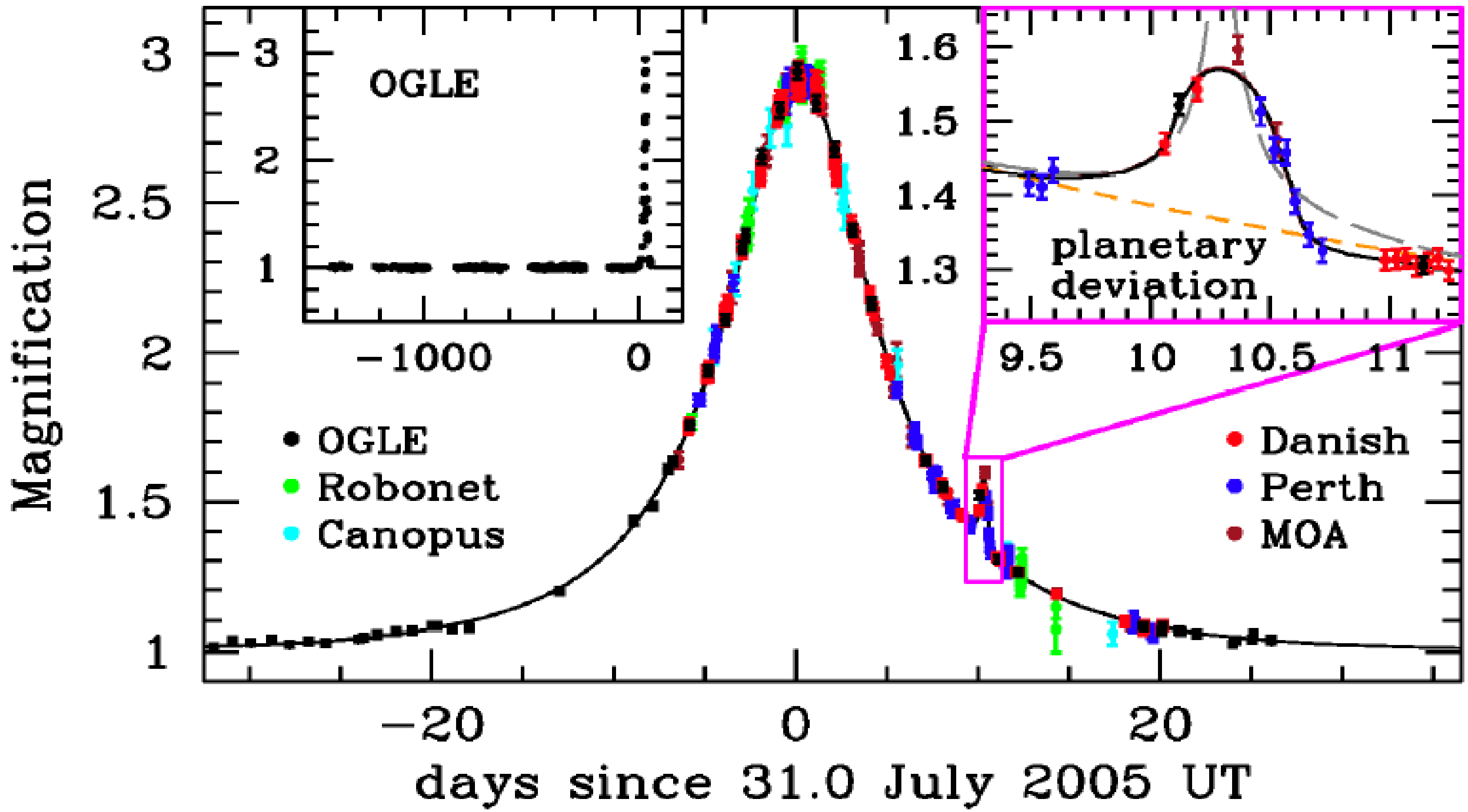
- Albrow et al. 1998 ApJ, 509, 687



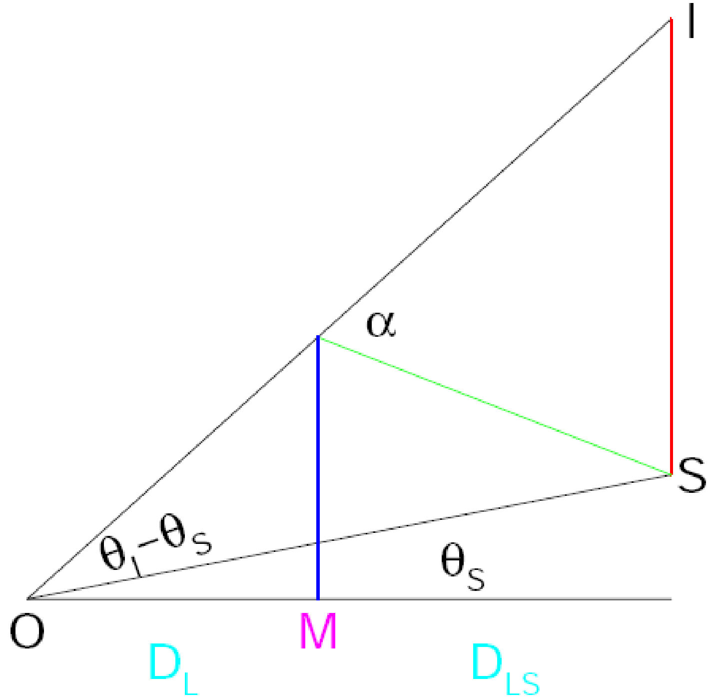


OGLE-2005-BLG-390

"Classical-Followup" Planetary Caustic



Beaulieu et al. 2006, Nature, 439, 437

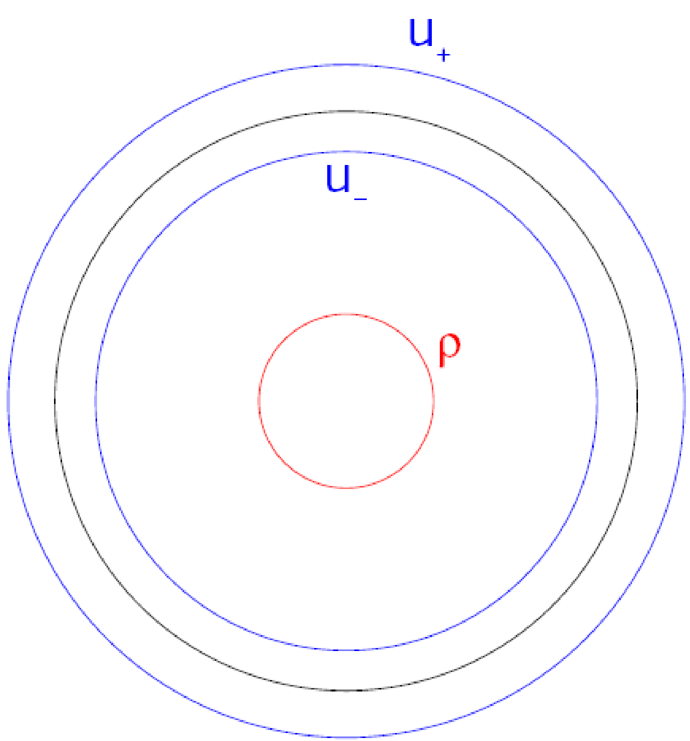


$$(\theta_I - \theta_S)D_S = \alpha D_{LS}$$

$$\alpha = 4GM/(D_L \theta_I c^2)$$

$$(\theta_I - \theta_S)\theta_I = \theta_E{}^2 = (4GM/c^2)(D_{LS}/D_L D_S)$$

$$\theta_I/\theta_E = [u +/- (u^2 + 4)^{1/2}]/2; \;\; u=\theta_S/\theta_E$$



Source Centered on Point Lens

$$A = \frac{\pi(u_+^2 - u_-^2)}{\pi\rho^2}, \qquad u_\pm = \frac{\rho \pm \sqrt{\rho^2+4}}{2}$$

$$A = \sqrt{1+\frac{4}{\rho^2}} \to 1 + \frac{2}{\rho^2}, \qquad \rho \equiv \frac{\theta_*}{\theta_{\rm E}}$$

Conjecture for Big Source on Planet Caustic

$$A_p = 2\left(\frac{\theta_{{\rm E},p}}{\theta_*}\right)^2$$

Plus Simple Timing Argument

$$\frac{t_p}{t_{\rm E}} = \frac{\theta_*}{\theta_{\rm E}}$$

Yields Mass-Ratio Estimate

$$q = \frac{M_p}{M} = \frac{\theta_{{\rm E},p}^2}{\theta_{\rm E}^2} = \frac{\theta_{{\rm E},p}^2}{\theta_*^2}\,\frac{\theta_*^2}{\theta_{\rm E}^2} = \frac{A_p}{2}\,\frac{t_p^2}{t_{\rm E}^2}$$

Mass-Ratio Estimate
a la Gould & Loeb

q=(A_p/2)(t_p/t_E)^2

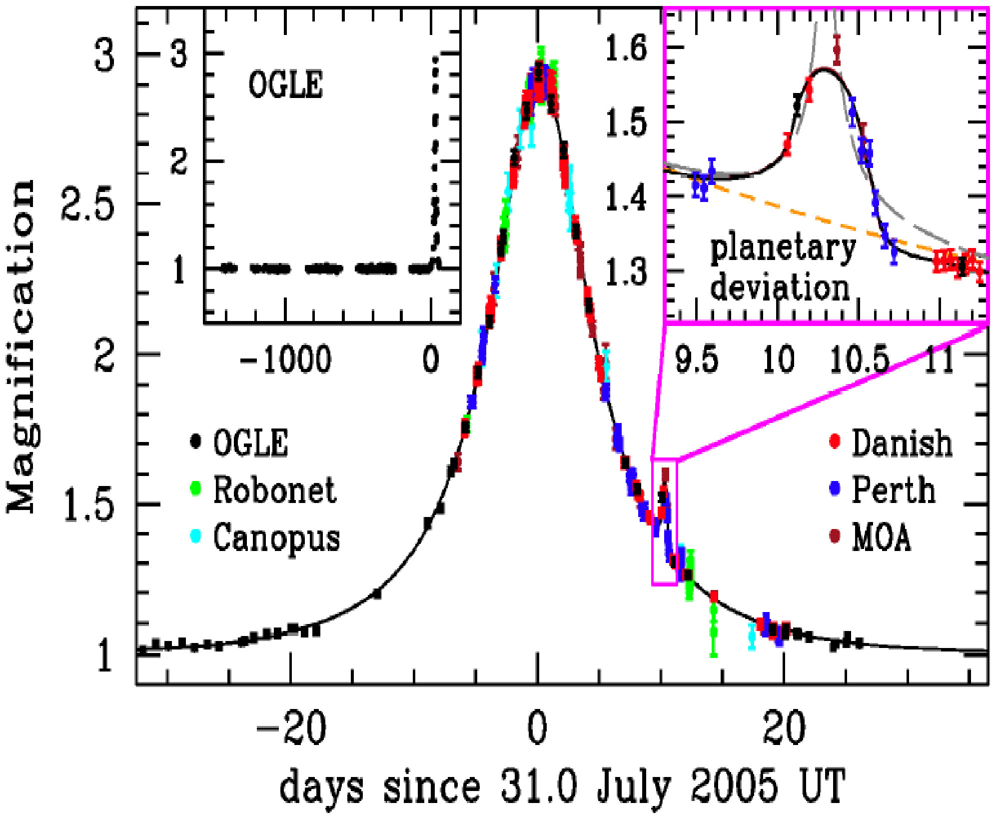
A_p = 0.2

t_p =0.3 day

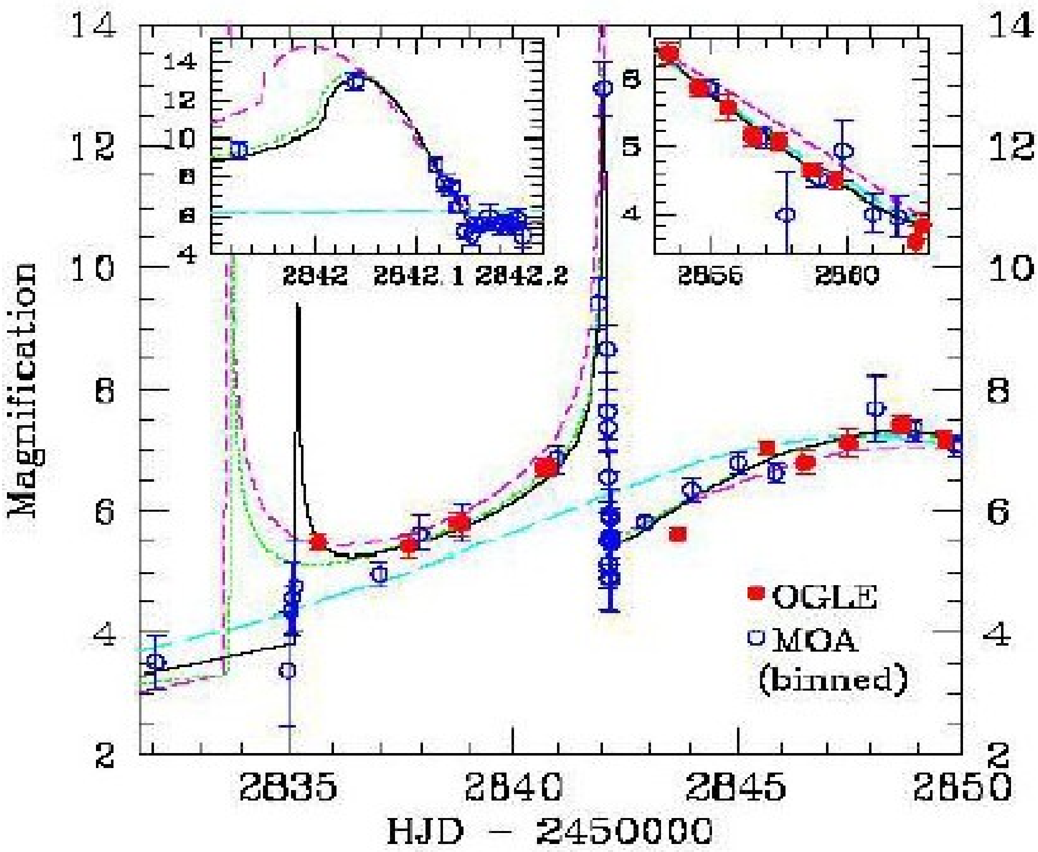
t_E=10 day

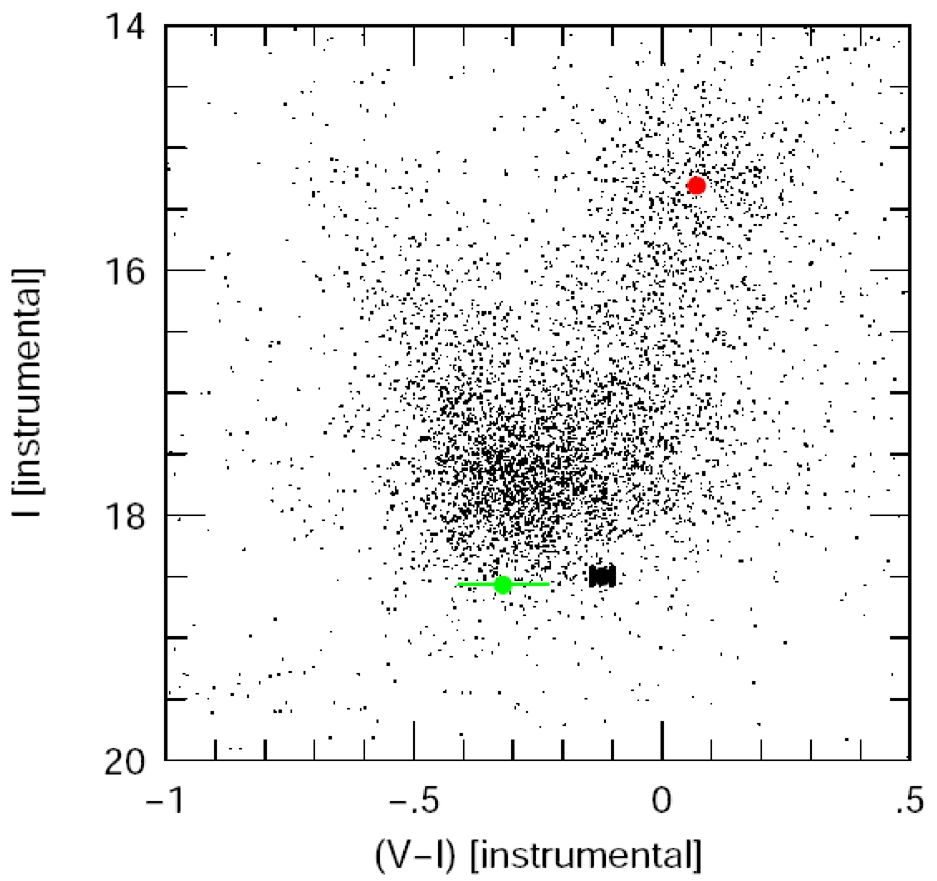
q=9e-5

q_actual = 8e-5

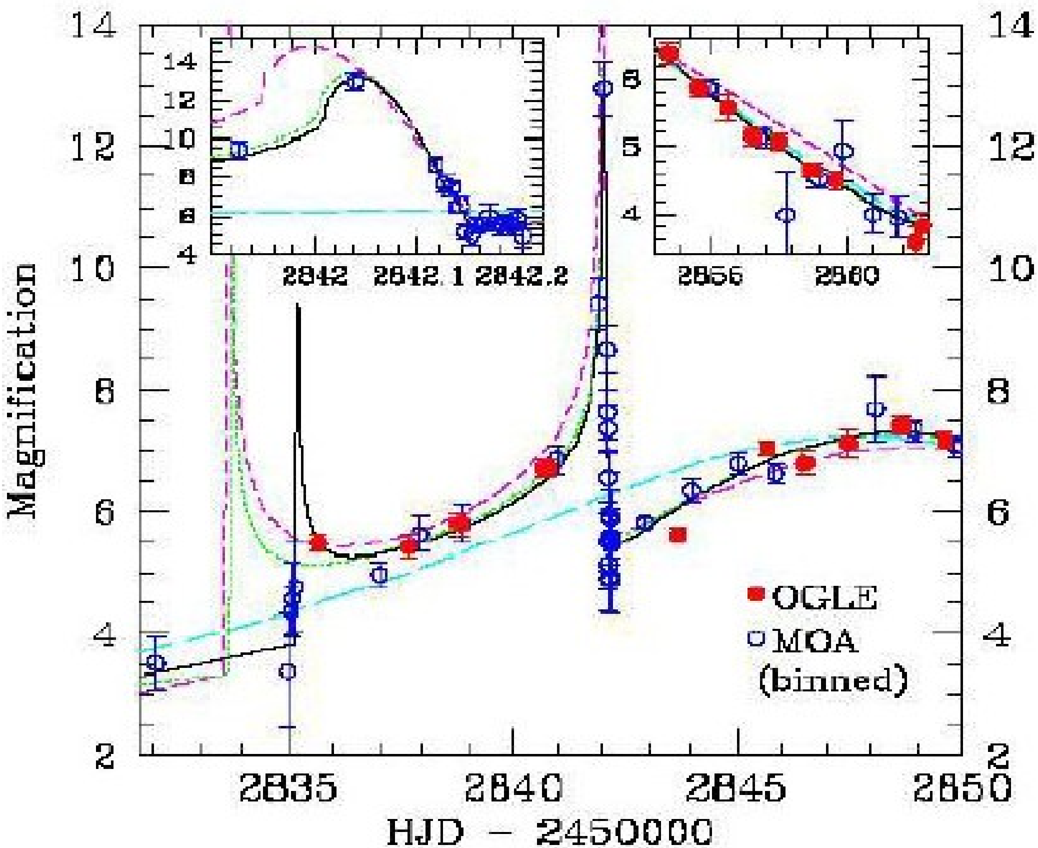


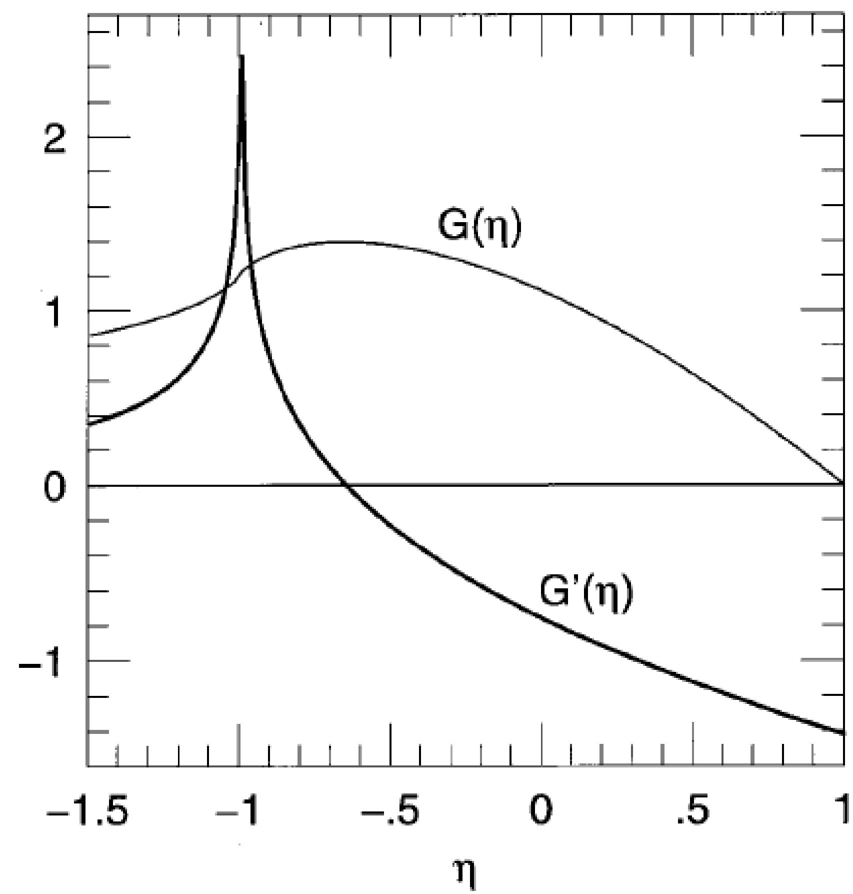
First Microlensing Planet
Pronounced Finite Source Effects





First Microlensing Planet
Perfect Fold Caustic Crossing





Mass-Ratio Estimate
a la Gould & Loeb

$\rho = 0.3/10 = 0.03$

$q = 8e\text{-}5$

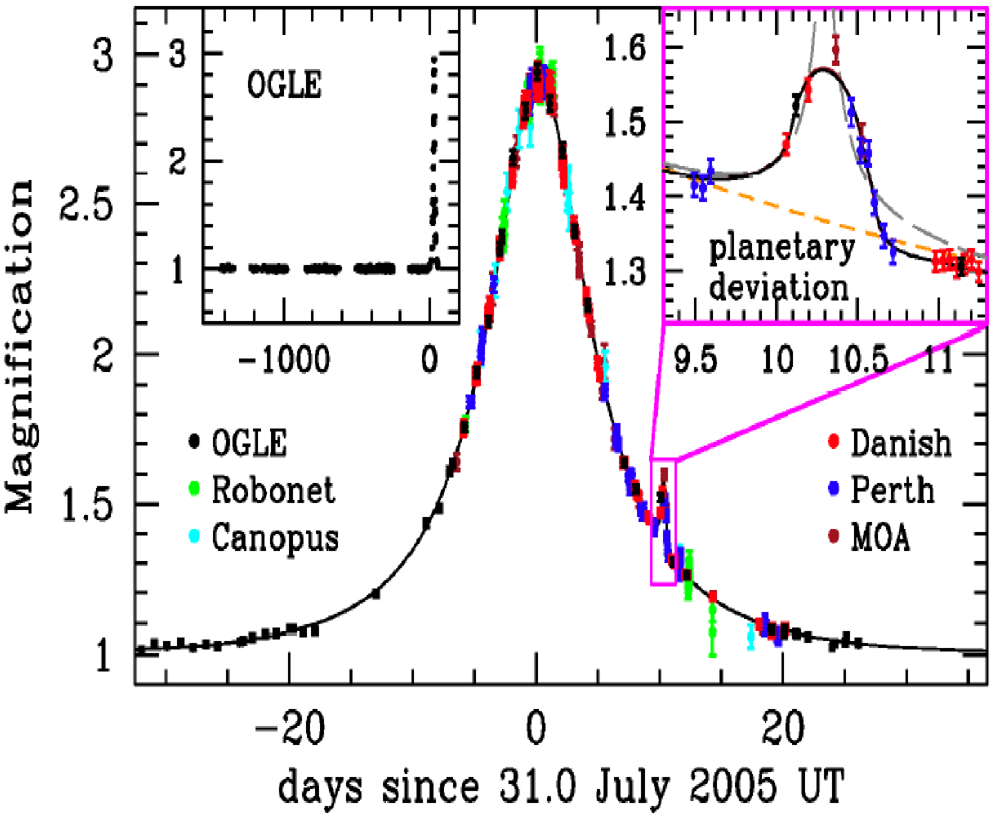
$\theta\_* = 5.73\ \mu as$

$\theta\_E = \theta\_*/\rho = 0.19$ mas

$M*\pi_rel = 4\ \mu$as-Msun

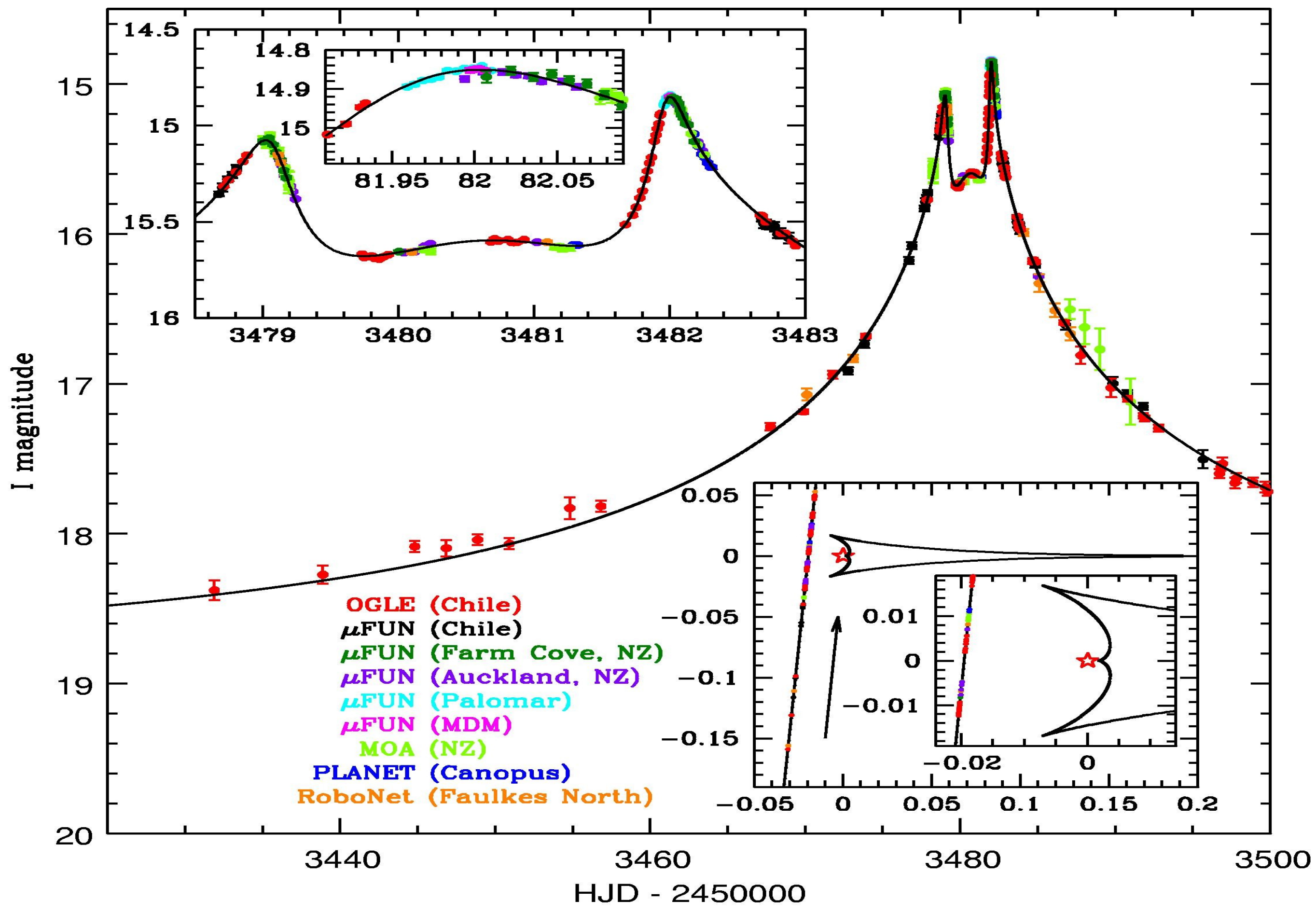
$\mu = \theta\_E/t\_E = 6.6$ mas/yr

$\theta_E^2 = \kappa\pi_{rel} M;\ \kappa = 8 mas/M_{sun}$



Second Microlensing Planet

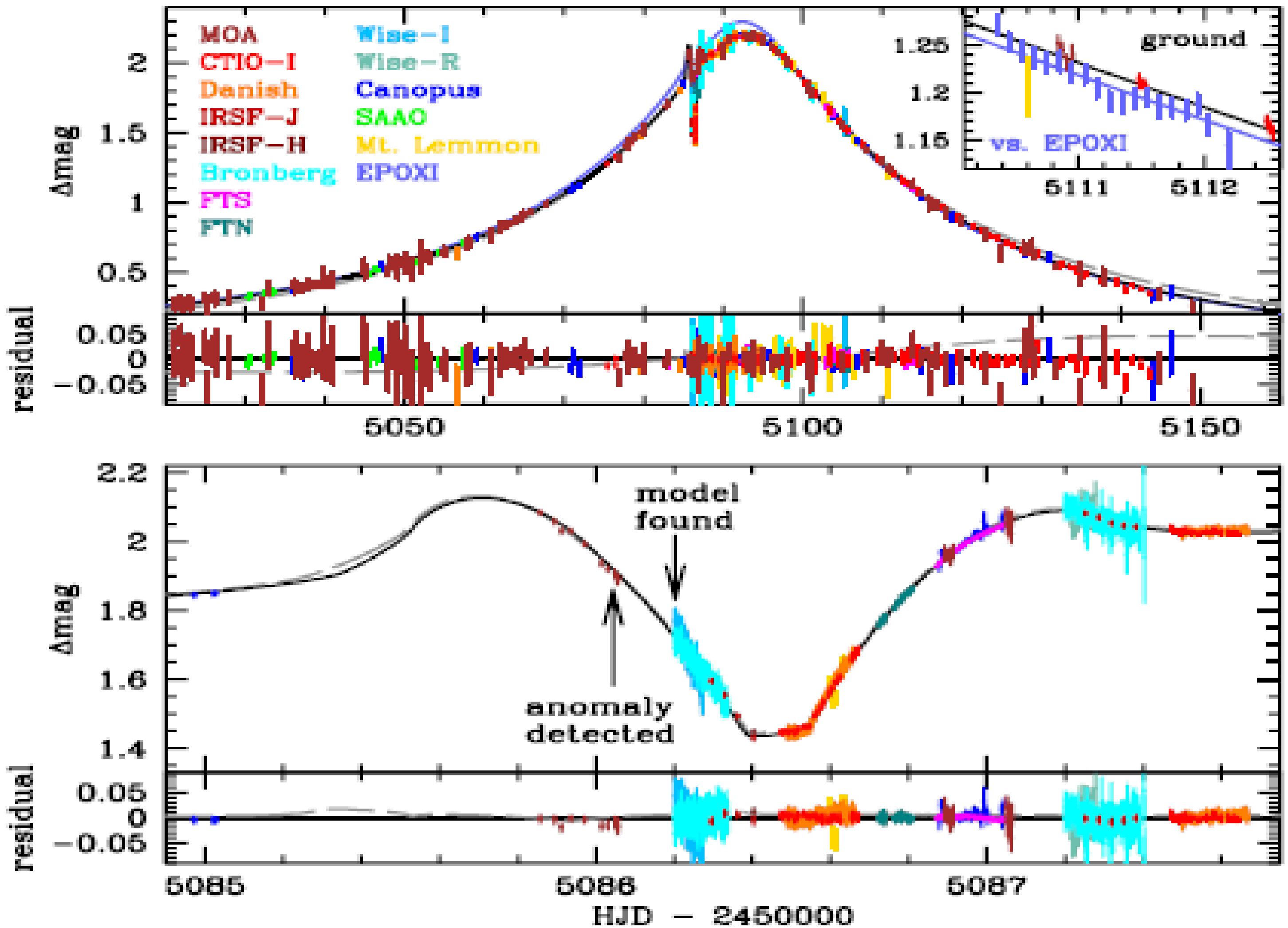
Weak Finite Source Effects



Udalski et al. 2005, ApJ, 628, L109

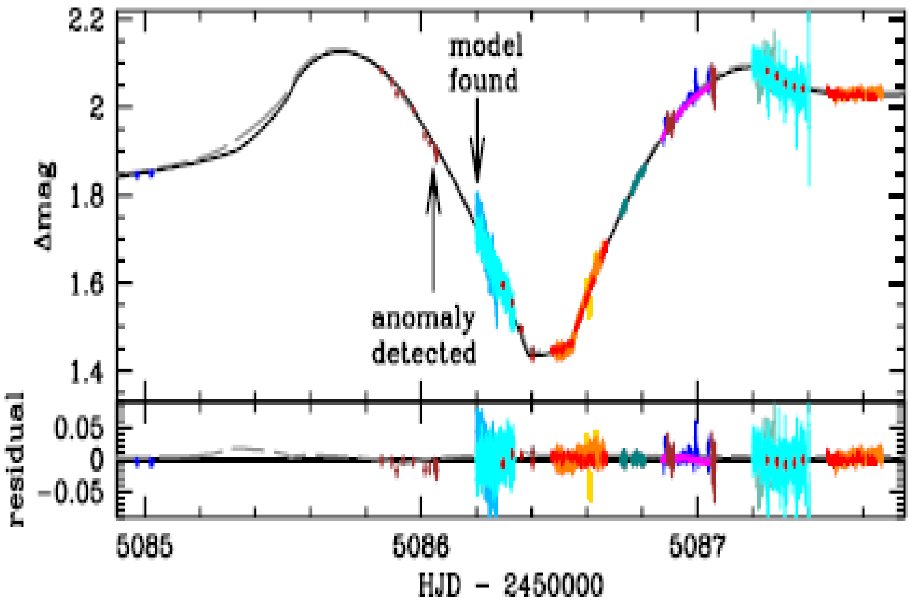
MOA-2009-BLG-266

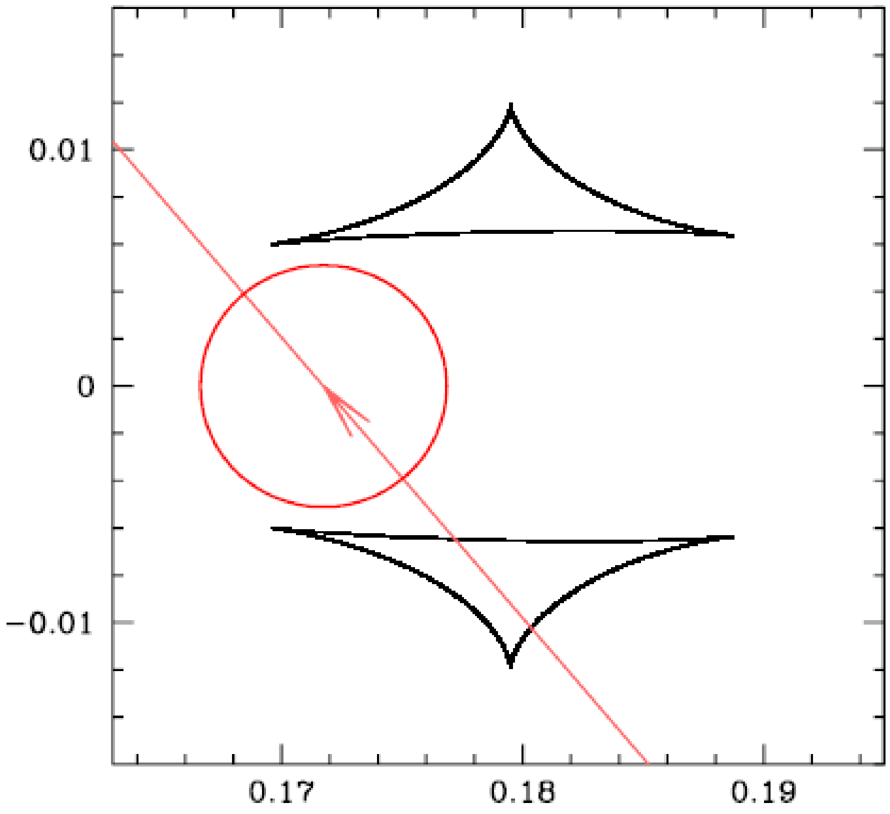
Parallax + Finite Source



Muraki et al. 2011, ApJ, 741, 22

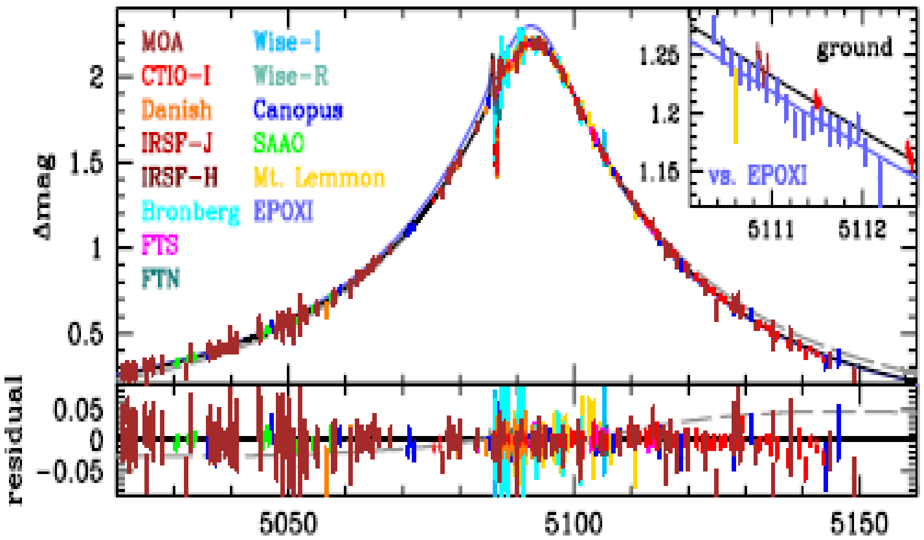
ρ Well-Measured from "Dip"

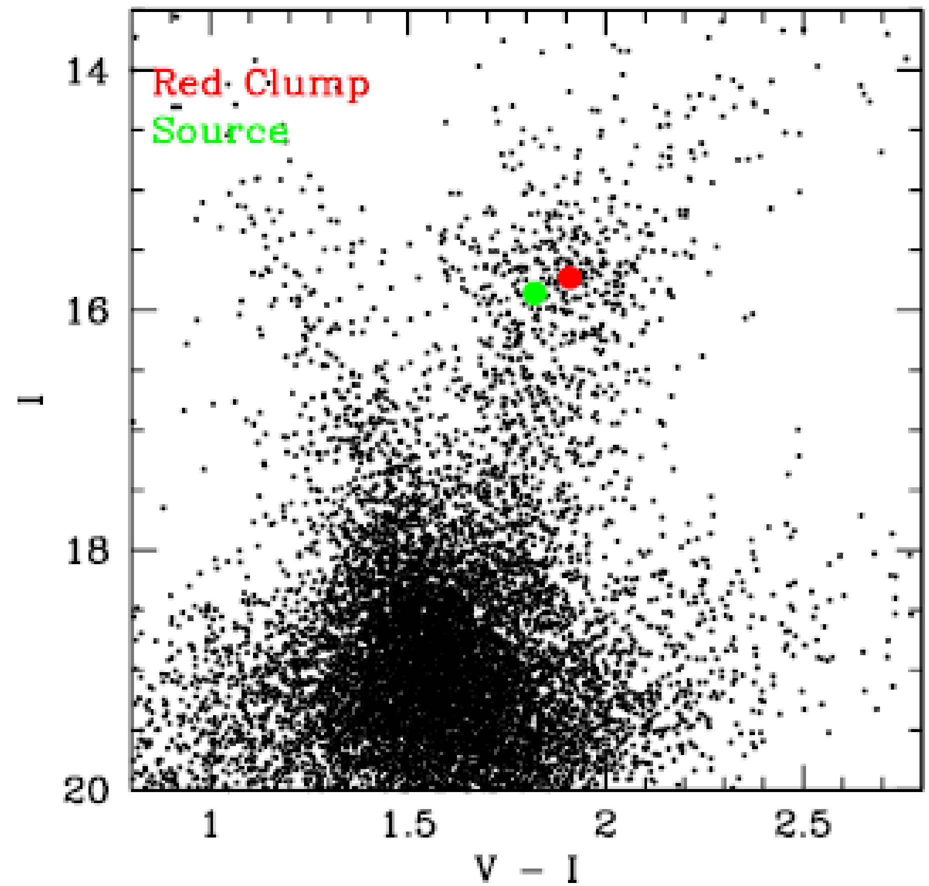




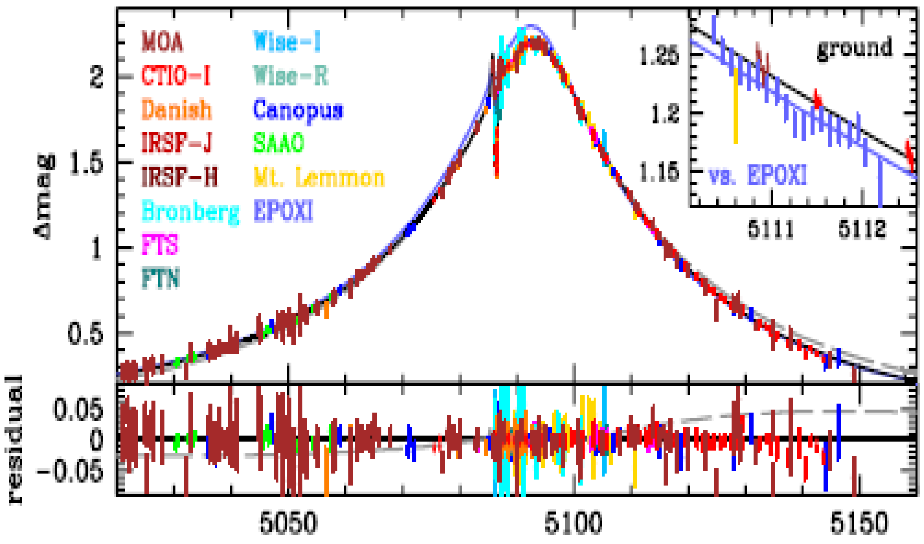
θ_* Well-measured from lightcurve

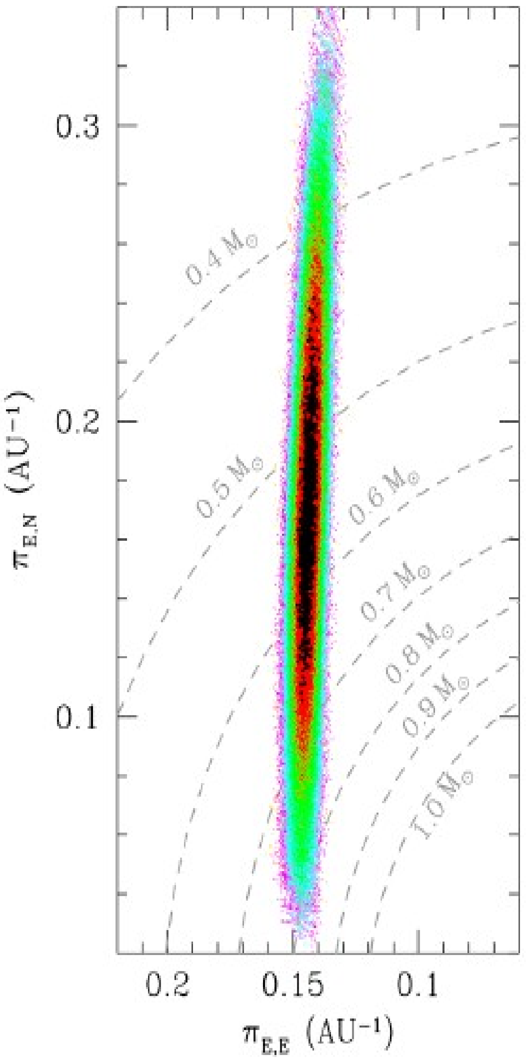
==> $\theta_E = \theta_*/\rho$





π_E semi-measured from lightcurve





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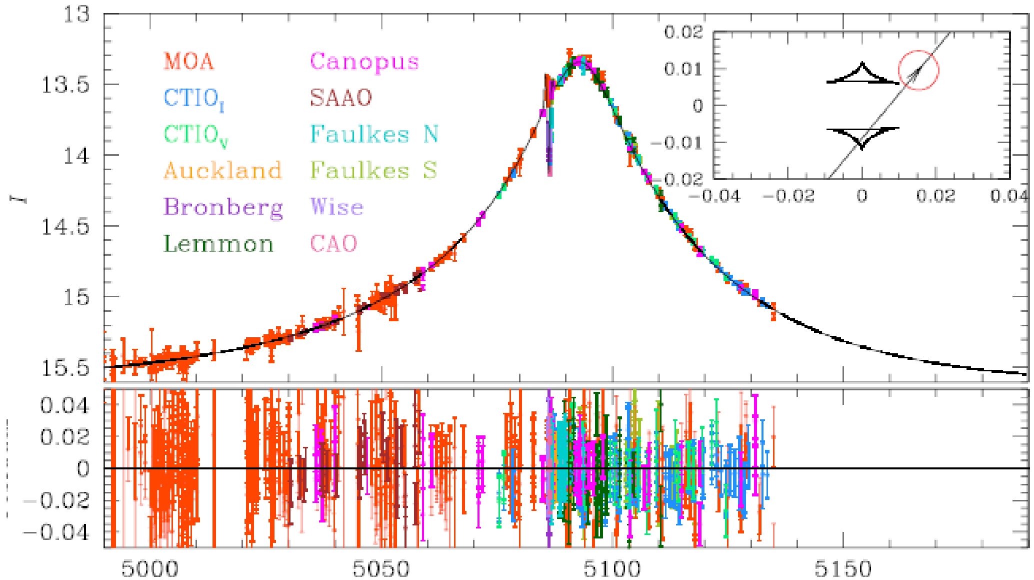

Main Parameters

$$t_0 = 5093.1 \qquad \text{[inspection]}$$

$$u_0 \simeq A_{\rm max}^{-1} = 10^{0.4(I_{\rm peak} - I_{\rm base})} = 0.132$$

$$[I_{\rm peak} = 13.35, \qquad I_{\rm base} = 15.55]$$

$$t_{\rm E,1} = \frac{t_{\rm eff}}{u_0} = \frac{t_0 - t_{1/2,-}}{u_0} = \frac{5093.1 - 5084.7}{0.132} = 64\,\text{day}$$

$$t_{1/2,-} = t[I = I_{\rm peak} + 2.5\log 2^{1/2}] = t[I = 13.73] = 5084.7$$

$$t_{\rm E,2} = t_0 - t_{\rm ring,-} = 5093 - 5036 = 57\,\text{day}$$

$$t_{\rm ring,-} = t[I = I_{\rm base} - 2.5\log(9/5)^{1/2}] = t[I = 15.23] = 5036$$

$$t_{\rm E} \to \frac{t_{\rm E,1} + t_{\rm E,2}}{2} = 60\,\text{day}$$

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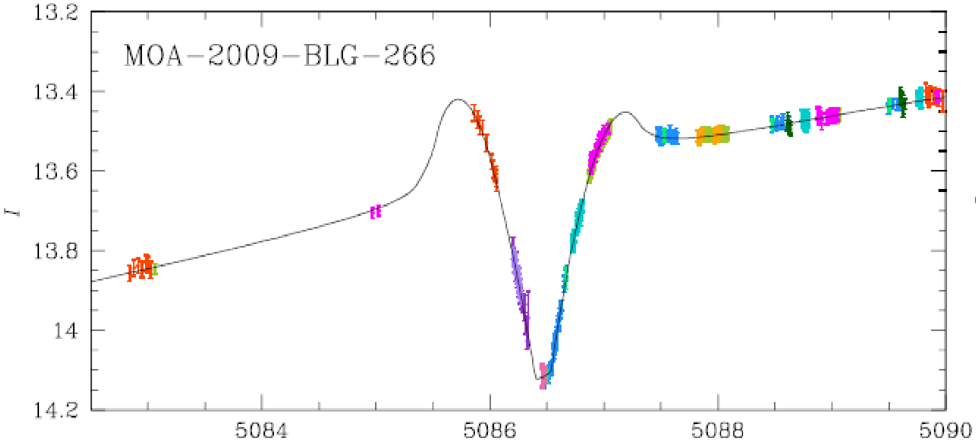
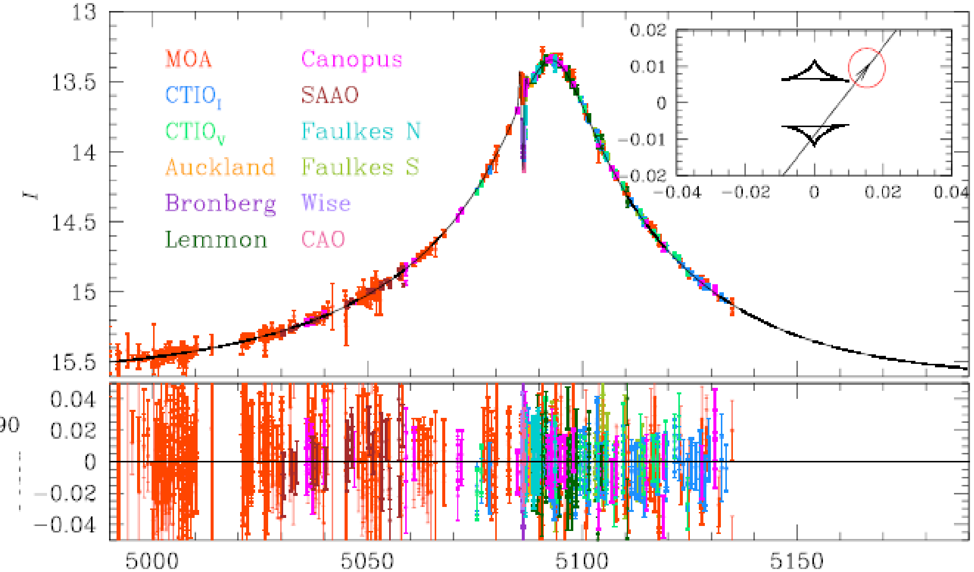




$$t_{0,\mathrm{planet}} = 5086.5 \qquad \tau_{\mathrm{planet}} = \frac{t_0 - t_{0,\mathrm{planet}}}{t_{\mathrm{E}}} = \frac{6.6}{60} = 0.11$$

$$u_{\mathrm{planet},1} = A_{\mathrm{planet}}^{-1} = 10^{0.4(I_{\mathrm{plan}} - I_{\mathrm{base}})} = 0.154 \qquad [I_{\mathrm{planet}} = 13.58]$$

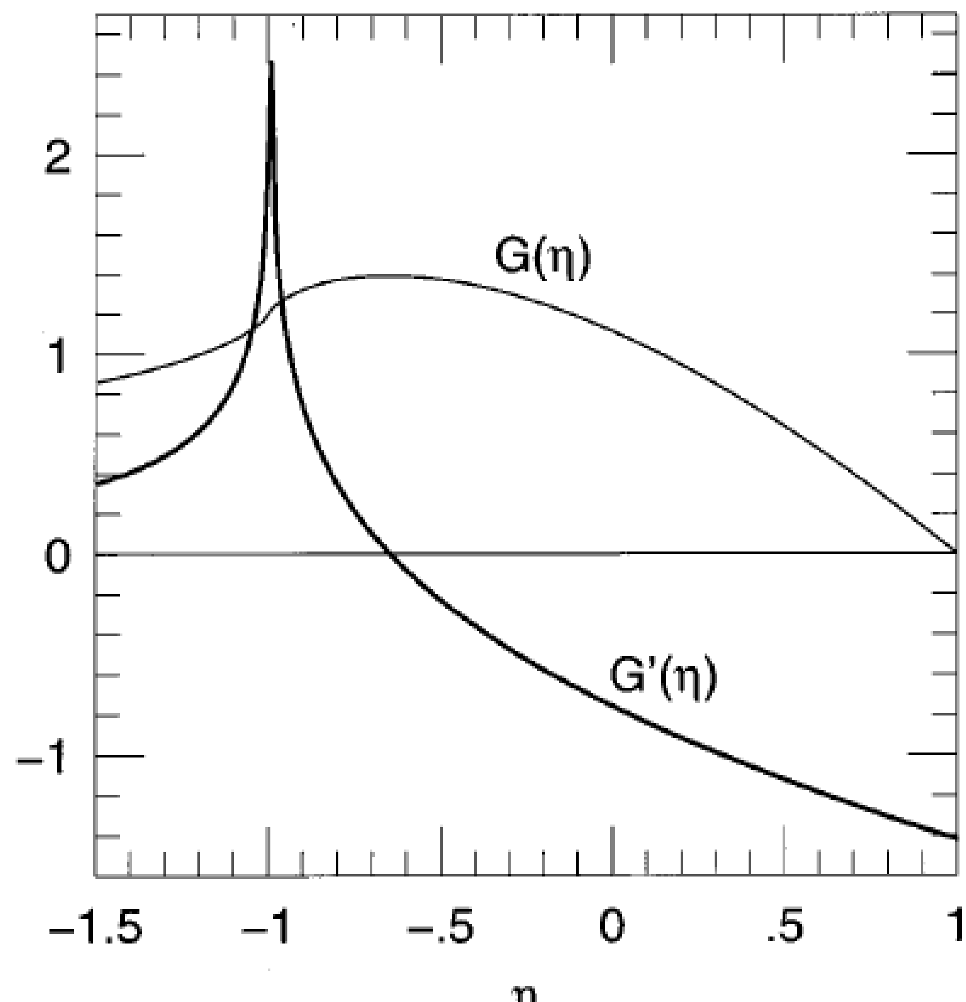
$$u_{\mathrm{planet},2} = \sqrt{u_0^2 + \tau_{\mathrm{planet}}^2} = 0.172$$

$$u_{\mathrm{planet}} = \frac{u_{\mathrm{planet},1} + u_{\mathrm{planet},2}}{2} = 0.163$$

$$s = \frac{-u_{\mathrm{planet}} + \sqrt{u_{\mathrm{planet}}^2 + 4}}{2} = 0.922$$

$$\alpha = \sin^{-1} \frac{u_0}{u_{\mathrm{planet}}} = 54^\circ$$

Generic Caustic Exit



Gould & Andronov 1999, ApJ, 516, 236

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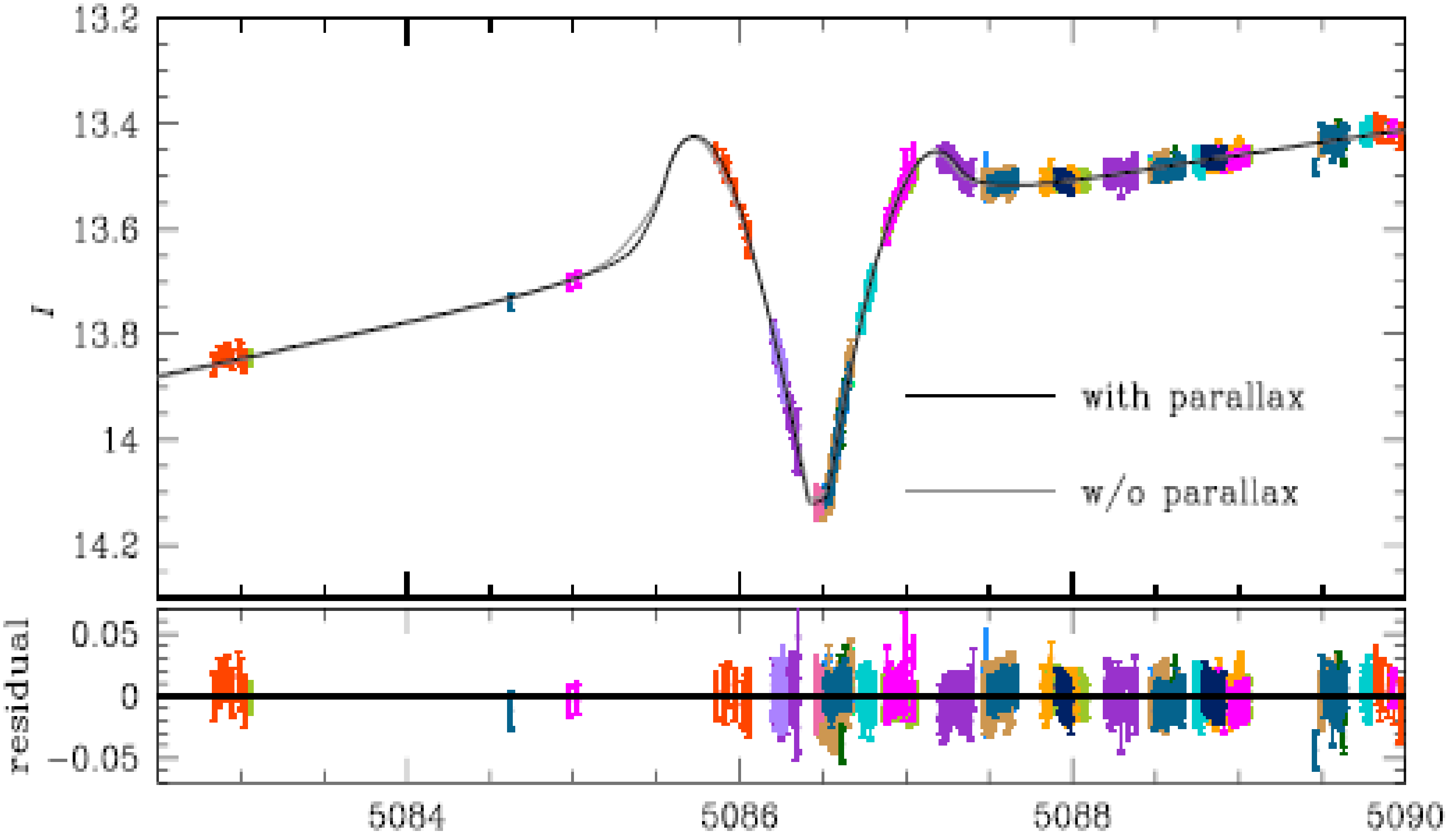

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Planet Parameters II: harder

$$t_{\rm cross,1} = \frac{t_{\rm planet-peak,1} - t_{\rm planet-trough,1}}{1.7} = 0.41\,{\rm day}$$

$$t_{\rm cc,1} = t_{\rm planet-peak,1} + 0.7 * t_{\rm cross,1} = 5085.98$$

$$t_{\rm planet-peak,1} = 5085.7, \qquad t_{\rm planet-trough,1} = 5086.4$$

$$t_{\rm cross,2} = \frac{t_{\rm planet-peak,2} - t_{\rm planet-trough,2}}{-1.7} = 0.38\,{\rm day}$$

$$t_{\rm cc,2} = t_{\rm planet-peak,2} - 0.7 * t_{\rm cross,2} = 5086.93$$

$$t_{\rm planet-peak,2} = 5087.2, \qquad t_{\rm planet-trough,1} = 5086.55$$

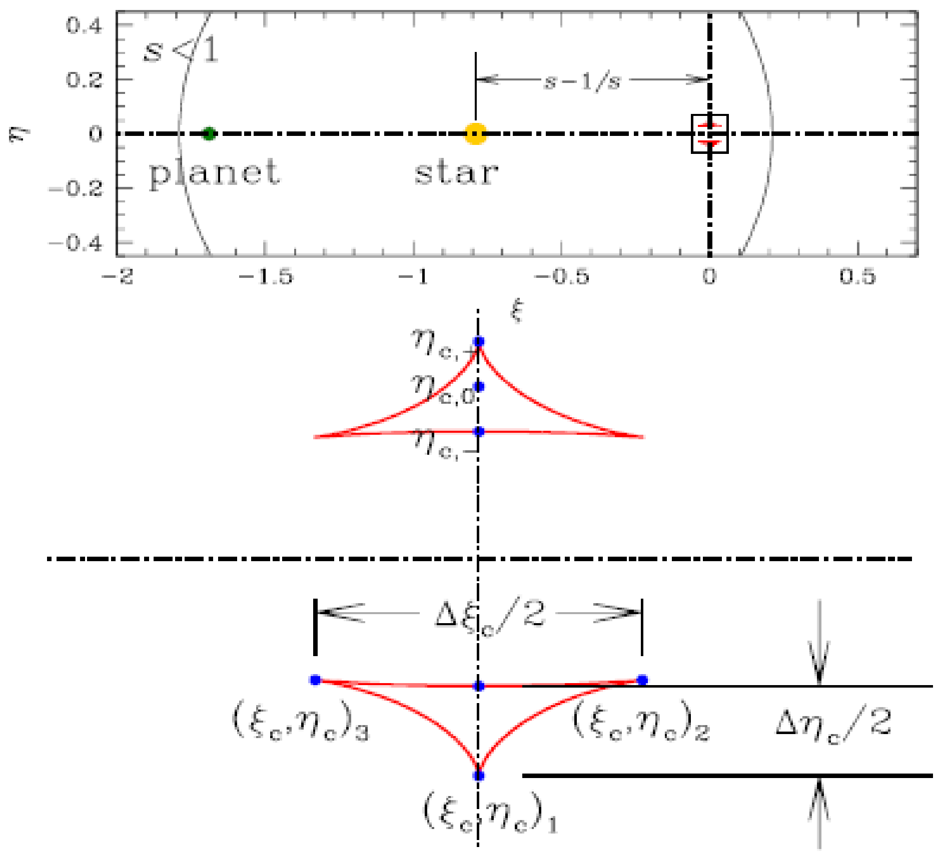
$$t_{\rm cross} = \frac{t_{\rm cross,1} + t_{\rm cross,2}}{2} = 0.397\,{\rm day}$$

$$\Delta u = \frac{t_{\rm cc,2} - t_{\rm cc,1}}{t_{\rm E}} \sin\alpha = 0.0128$$

$$\Delta u = 4\sqrt{\frac{q u_{\rm planet}}{s}} \Rightarrow q = \frac{s}{u_{\rm planet}} \left(\frac{\Delta u}{4}\right)^2 = 5.8 \times 10^{-5}$$

$$t_* = t_{\rm cross} \sin\alpha = 0.32\,{\rm day}, \qquad \rho = \frac{t_*}{t_{\rm E}} = 5.3 \times 10^{-3}$$

Minor Image Analytic Formulae



$$\eta_{c,-} = 2q^{1/2}(1-s^2)^{1/2}/s$$

Han 2006, ApJ, 638, 1080

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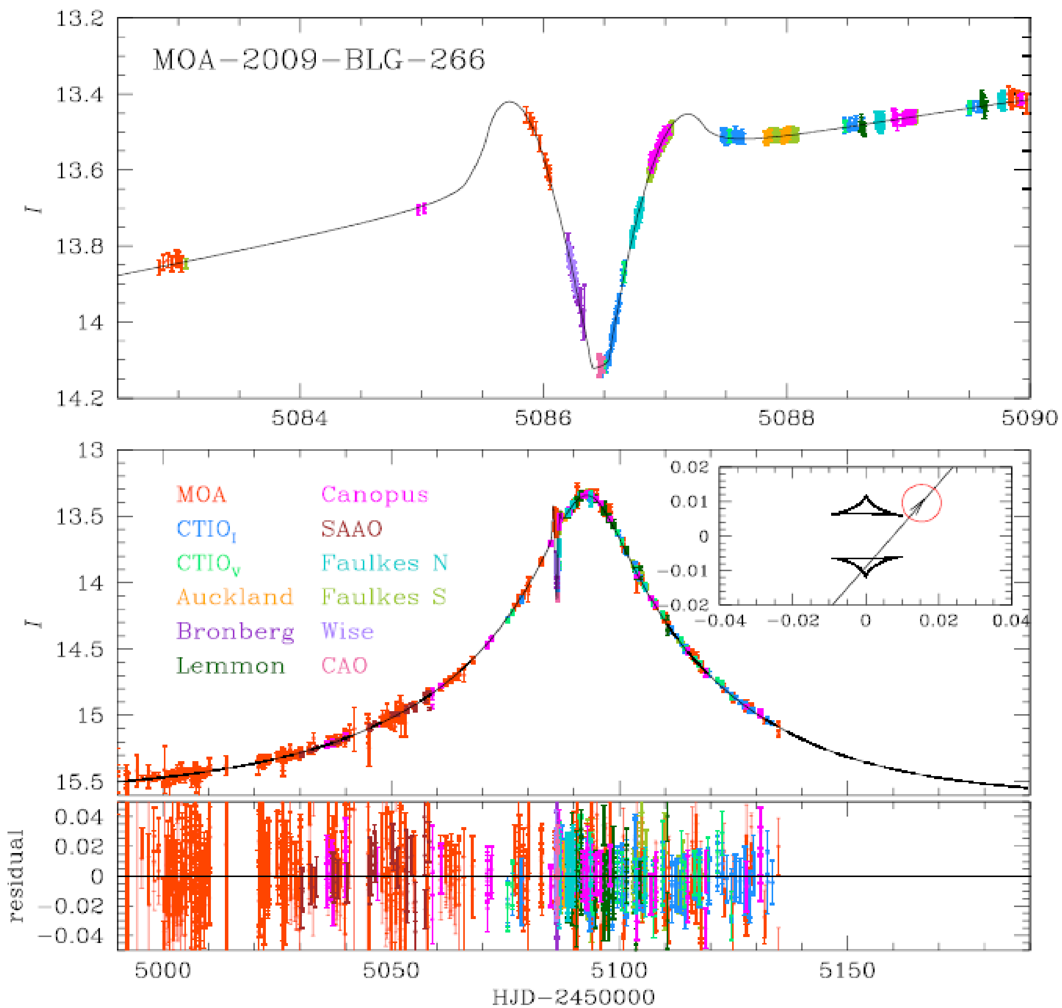
Minor Image Planetary Caustic



Preliminary Model (Cheongho Han)

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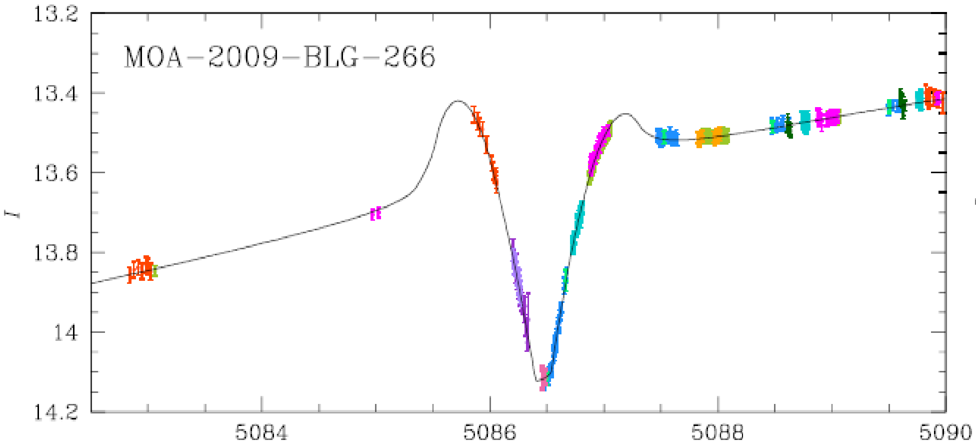


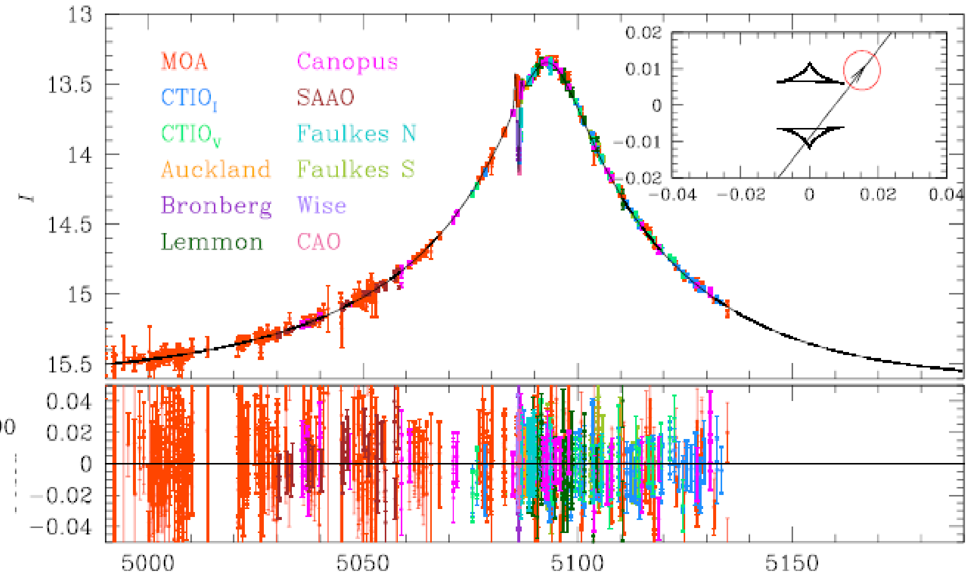


TABLE 1

MB09266: Eye vs. Computer

Parameter	Eye	Computer
t_0	5093.1	5093.07
u_0	0.13	0.13
t_E	60 d	60.2 d
q	5.8×10^{-5}	5.4×10^{-5}
s	0.922	0.914
α	54°	51°
ρ	5.3×10^{-3}	5.3×10^{-3}

Minor Image Test

$$\frac{A_{\rm trough}}{A_{\rm planet}} = 10^{0.4(I_{\rm planet} - I_{\rm trough})}$$

$$= 10^{0.4(13.58-14.02)} = 0.667$$

$$\frac{A_{\rm planet}+1}{2A_{\rm planet}} = 0.657$$

Planet Lenses Often Have Parallax

9 Features & 9 Parameters

9 Features	9 Parameters
3 Point-Lens	t_0, u_0, t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q
Width of Caustic Cr.	$t_* = \rho * t_E$
Symmetric Distortion	$\pi_{E,perp}$
Anti-symmetric Dist.	$\pi_{E,parallel}$

Point-lens magnfication

Start: Binary-Lens Equation

$$\mathbf{u} - \mathbf{y} = -\frac{\mathbf{y} - \mathbf{y}_L}{|\mathbf{y} - \mathbf{y}_L|^2}$$

$$\mathbf{y}_L = 0 \rightarrow \mathbf{u} - \mathbf{y} = -\frac{\mathbf{y}}{y^2} \Longrightarrow u - y = -\frac{1}{y}$$

$$\Longrightarrow (y-u)y = 1 \Longrightarrow (\theta_I - \theta_S)\theta_I = \theta_{\mathrm{E}}^2$$

$$\mathbf{u} = \mathbf{y} - \sum_i \epsilon_i \frac{\mathbf{y} - \mathbf{y}_{m,i}}{|\mathbf{y} - \mathbf{y}_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{\mathrm{tot}}}$$

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + iu_2 \qquad z \equiv y_1 + iy_2$$

Why is this a Fifth-Order Equation?

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + iu_2; \qquad z \equiv y_1 + iy_2$$

$$z = \zeta + \frac{\epsilon_1}{\bar{z} - \bar{z}_1} + \frac{\epsilon_2}{\bar{z} - \bar{z}_2}$$

$$\bar{z} = \bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2}$$

$$(z - \zeta)(\bar{z} - \bar{z}_1)(\bar{z} - \bar{z}_2) = \epsilon_1(\bar{z} - \bar{z}_2) + \epsilon_2(\bar{z} - \bar{z}_1)$$

$$(z - \zeta)\left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_1\right)\left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_2\right)$$

$$= \left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_2\right)\epsilon_1 + \left(\bar{\zeta} + \frac{\epsilon_1}{z - z_1} + \frac{\epsilon_2}{z - z_2} - \bar{z}_1\right)\epsilon_2$$

Magnification (A): For each image, i

1) Check that it solves lens equation

2) Calculate A_i from determinant

$$\partial\zeta_i = \sum_k \frac{\epsilon_k}{(\bar{z} - \bar{z}_k)^2}$$

$$A_i = \frac{1}{1 - |\partial\zeta_i|^2}$$

$$A = \sum_i |A_i|$$

Quadrupole/Hexadecapole

Pejcha & Heyrovsky (2009)

Gould (2008)

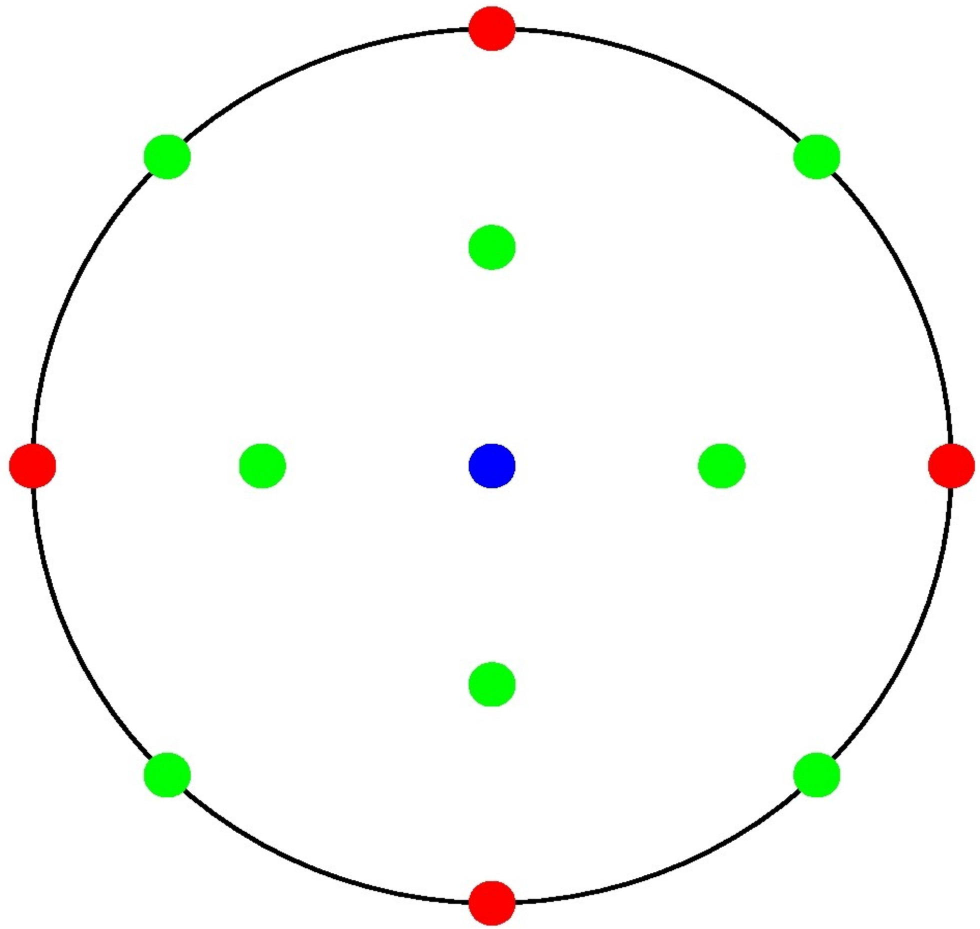
Pure gradient: Monopole (pt lens)

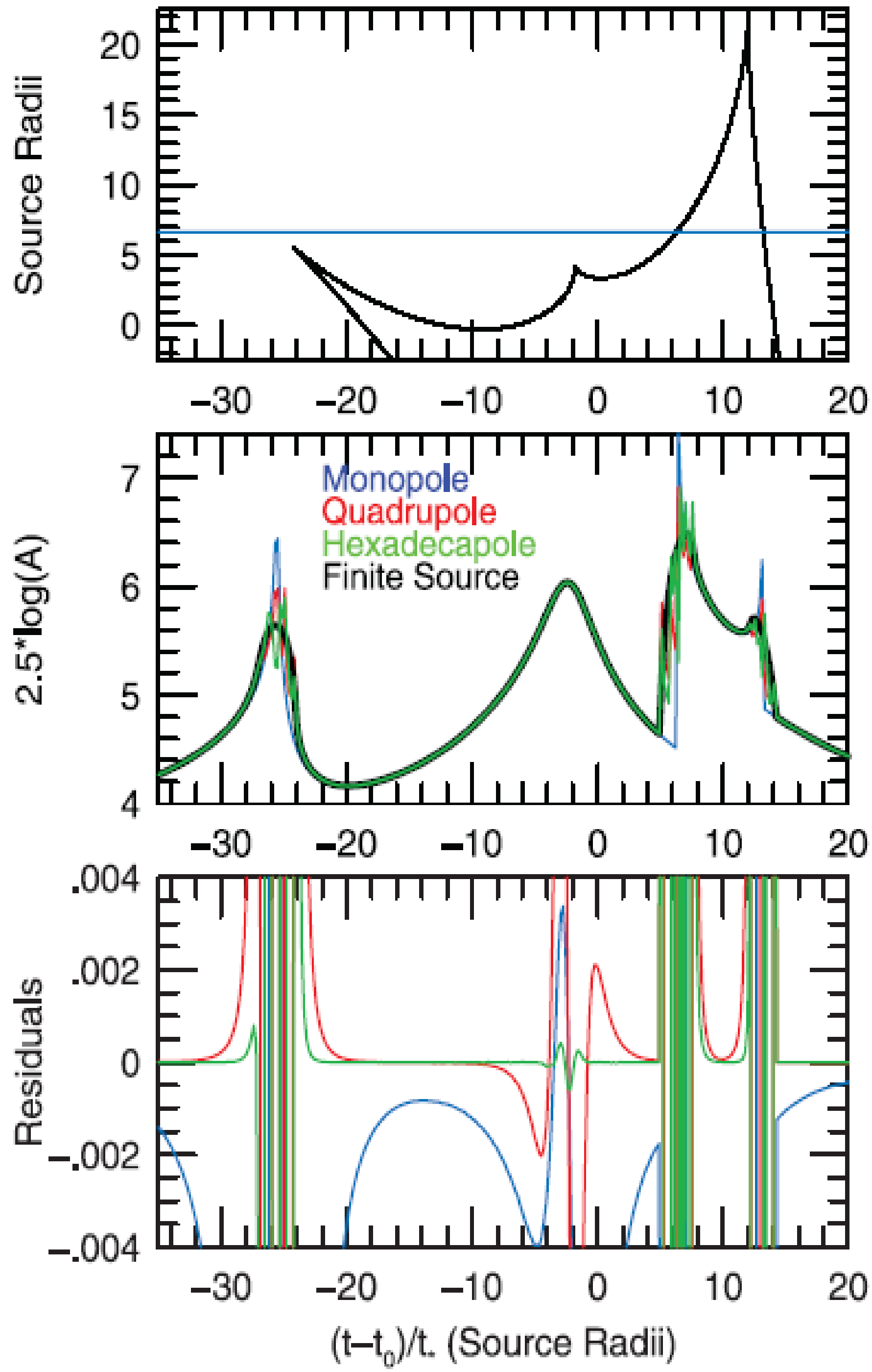
Mild curvature: Quadrupole

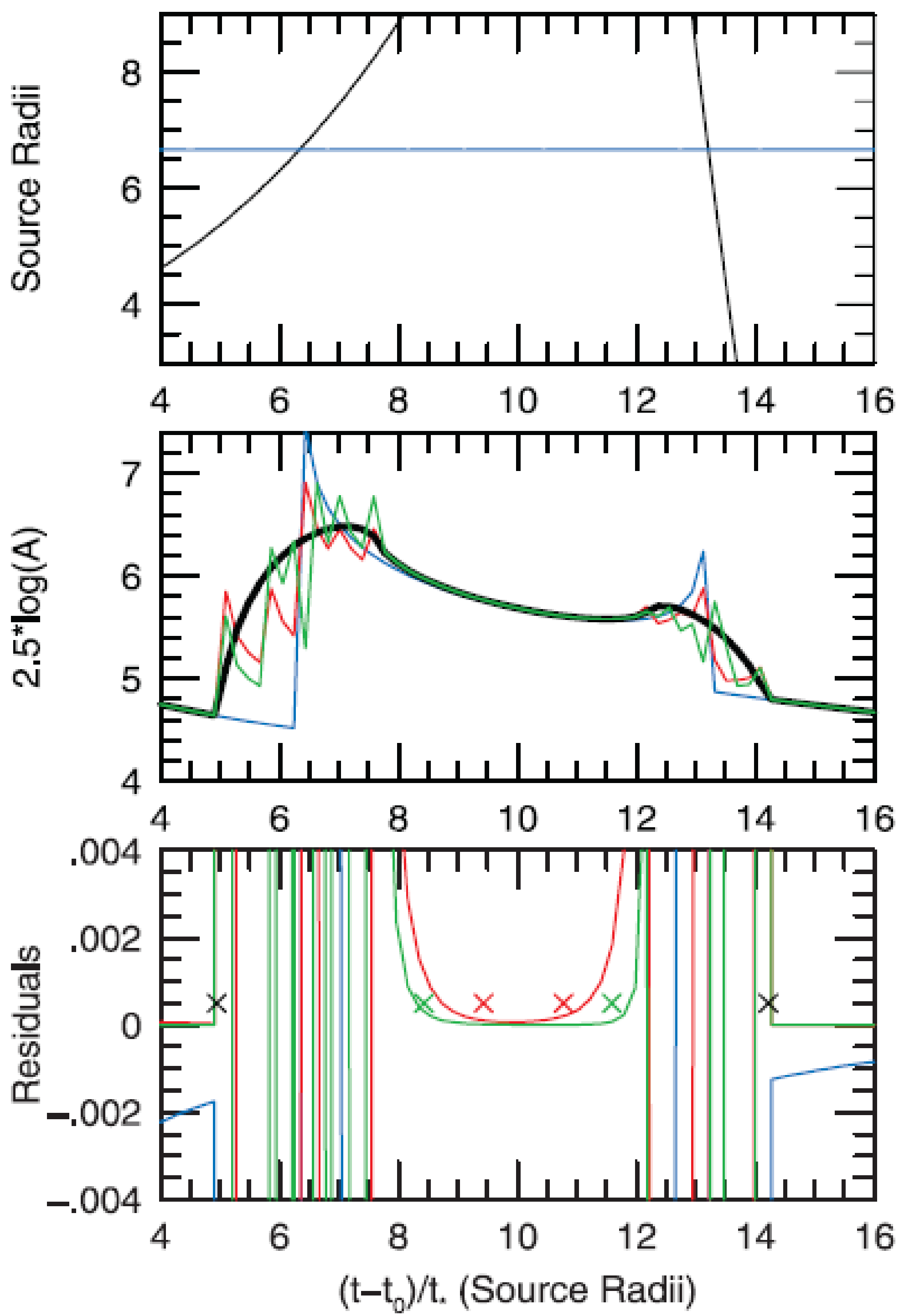
Stronger curvature: Hexadecapole

Extreme curvature: New Method

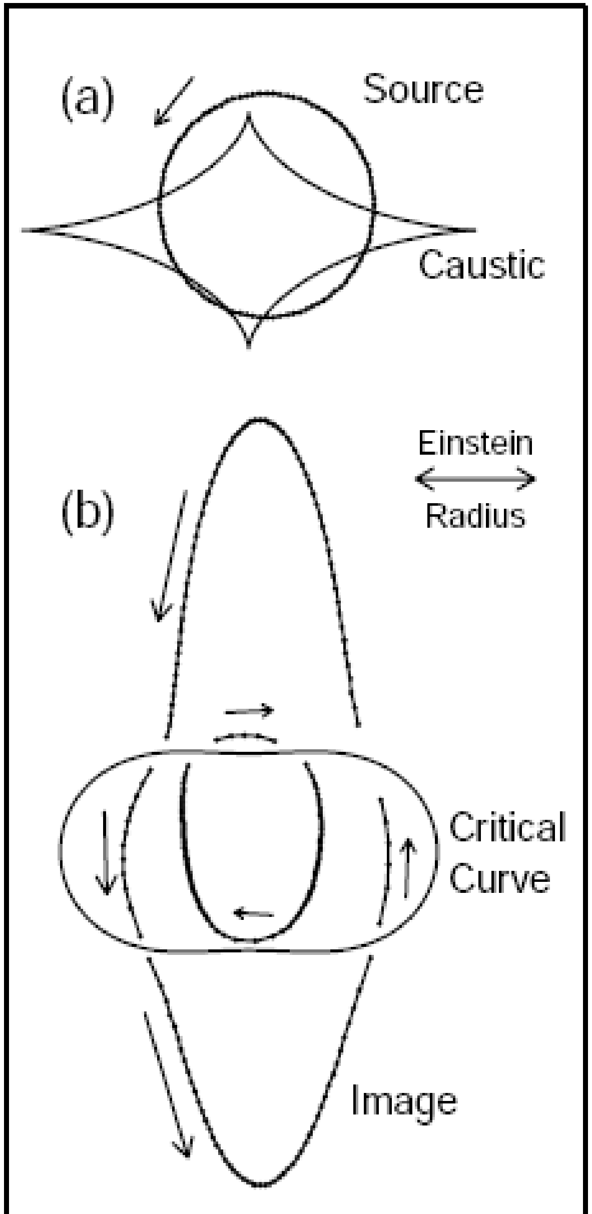
Monopole/Quadrupole/Hexadecapole




Source Radii
2.5*log(A)
Monopole
Quadrupole
Hexadecapole
Finite Source
Residuals
(t−t$_0$)/t. (Source Radii)


Source Radii
2.5*log(A)
Residuals
(t−t0)/t. (Source Radii)

Contour Integration: Gould & Gaucherel (1997)



$$A = \sum_{i=1}^{n} \sum_{j'} p_j (\mathbf{u}_{i-1,j} \times \mathbf{u}_{i,j}) \Big/ \sum_{i=1}^{n} \mathbf{s}_{i-1} \times \mathbf{s}_i \,,$$

Contour Integration

Fastest INDIVIDUAL FS calculation

Disadvantage 1: Limb Darkening -> many cont.

Disadvantage 2: Cusp lens-solver hang-ups

Neither fatal (see below)

Major improvements from Bozza (2010)

MNRAS 408 2188 (code not public)

Inverse Ray Shooting (General)

$$u = y - \sum_i \epsilon_i \frac{y - y_{m,i}}{|y - y_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{\rm tot}}$$

do i=1,n

  pick: y_i

    image plane point

  calculate: u_i

    source plane point

  store (u_i <--> y_i)

enddo

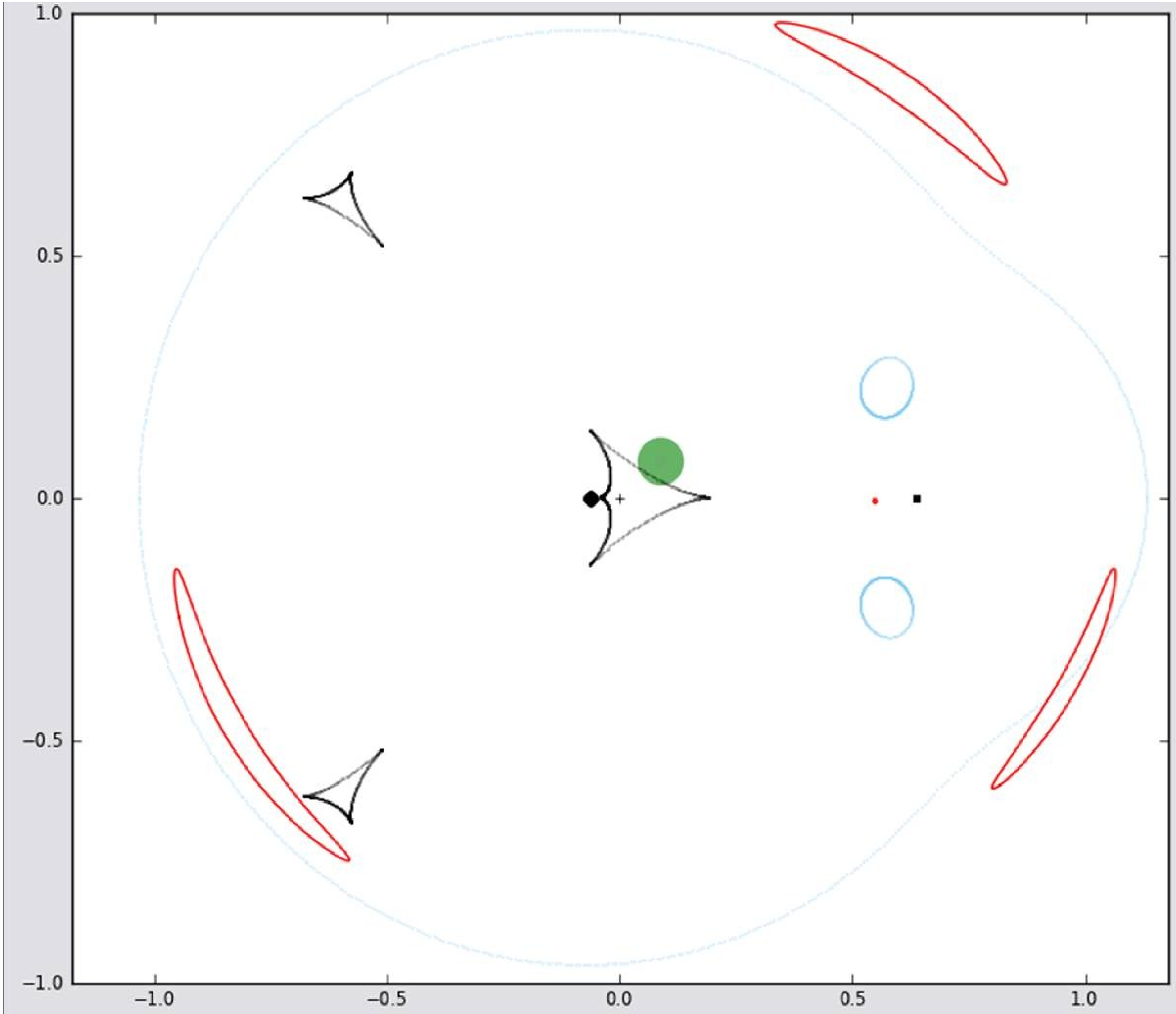
Pick source boundary

Examine each pt u_i

  in: weight by LD

  out: discard

Sum up weights


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

Inverse Ray Shooting

- Advantages
  - Automatically includes LD
  - Always works
- Disadvantage
  - Very expensive to shoot lens plane

IRS 1: Map-Making

Dong et al. 2006 ApJ 642 842

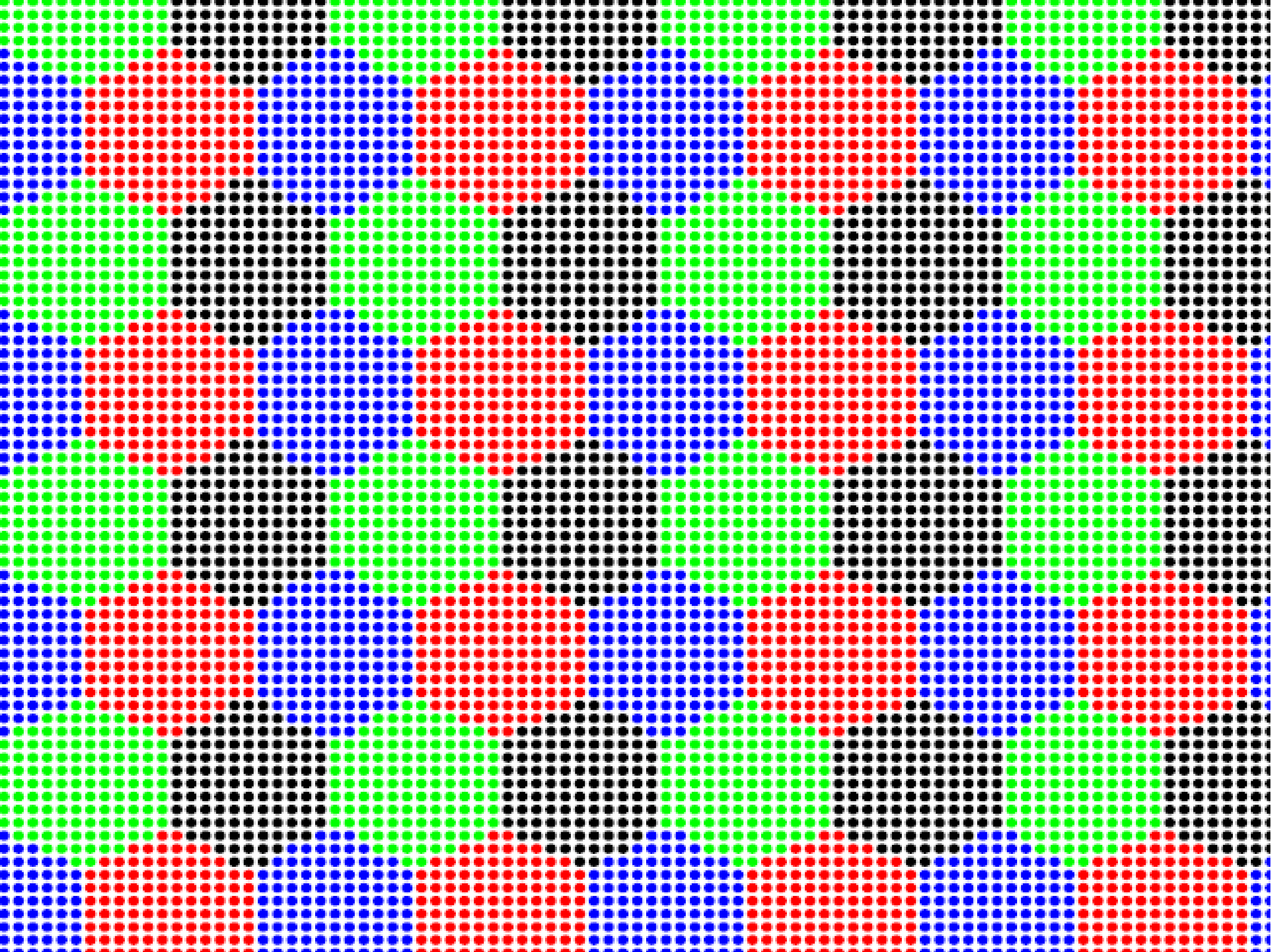
Shoot entire annulus relevant to event

Store rays & hex-tiles on source plane

Use hex-tiles for interior, rays for edge

All (s,q) light curves use ONE map

Disadvantage: requires fixed "s"



IRS 2A: Loop Linking

Dong et al. 2006 ApJ 642 842

Make contour (as in Gould+Gaucherel)

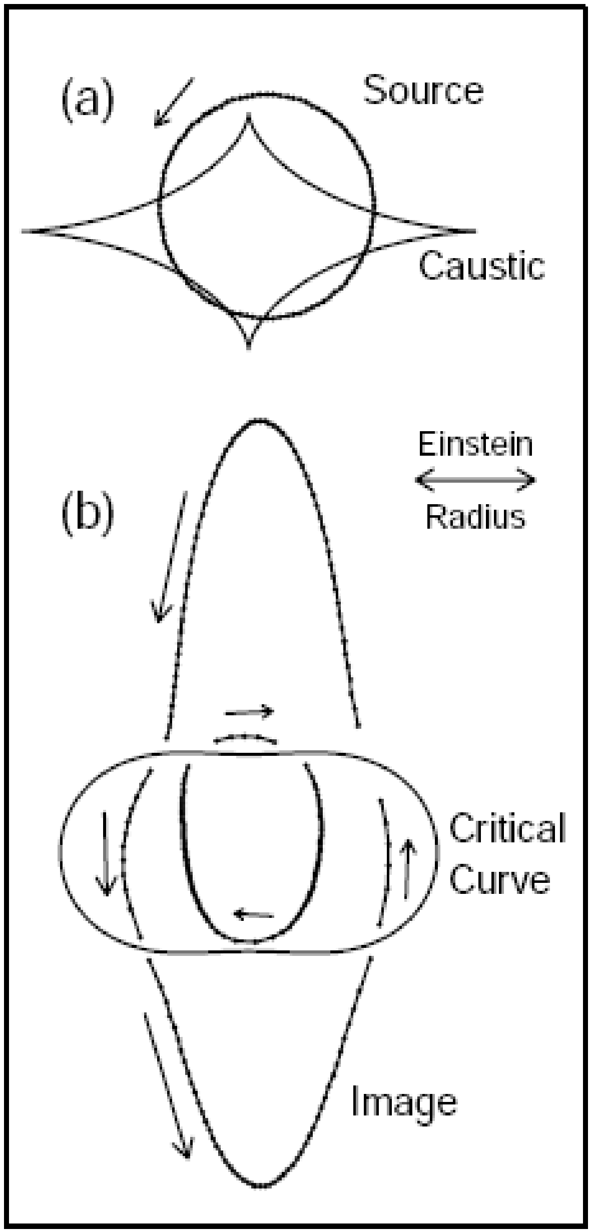
Except slightly bigger

Shoot rays within images (IRS general)

Advantage: avoids contour problems

Disadvantage: costs more than contour

Contour Integration: Gould & Gaucherel (1997)



$$A = \sum_{i=1}^{n} \sum_{j'} p_j (u_{i-1,j} \times u_{i,j}) \Big/ \sum_{i=1}^{n} s_{i-1} \times s_i \, ,$$

IRS 2B: Adaptive Images

Bennett 2010, ApJ, 716, 1408

Begin with image centers (point lens)

Expand coverage to source boundary

Radial coord, boundary-sensitive integ.

Advantage: precision with fewer rays

Disadvantage: costs more than contour

Linear Limb Darkening

$$S(\theta) = \frac{3F}{(3-u)\pi\theta_*^2}\left[1 - u\left(1 - \sqrt{1 - \frac{\theta^2}{\theta_*^2}}\right)\right]$$

$$S(\theta) = \frac{F}{\pi\theta_*^2}\left[1 - \Gamma\left(1 - \frac{3}{2}\sqrt{1 - \frac{\theta^2}{\theta_*^2}}\right)\right]$$

$$\Gamma = \frac{2u}{3-u}$$

$$\langle r^n \rangle = \frac{\rho^n}{n/2+1}(1 - \alpha_n\Gamma); \qquad \alpha_n = 1 - \frac{(3/2)!(1+n/2)!}{(3/2+n/2)!}$$

$$\langle r^2 \rangle = \frac{\rho^2}{2}\left(1 - \frac{1}{5}\Gamma\right) \qquad \langle r^4 \rangle = \frac{\rho^4}{3}\left(1 - \frac{11}{35}\Gamma\right)$$

Limb Darkening Applications

Point Lens

$$A_{\rm max} = \langle r^{-1} \rangle = \frac{2}{\rho}\left[1 + \left(\frac{3\pi}{8} - 1\right)\Gamma\right]$$

Hexadecapole

$$A_{\rm finite} = A_0 \langle r^0 \rangle + A_2 \langle r^2 \rangle + A_4 \langle r^4 \rangle$$

$$= A_0 + \frac{A_2 \rho^2}{2}\left(1 - \frac{1}{5}\Gamma\right) + \frac{A_4 \rho^4}{2}\left(1 - \frac{11}{35}\Gamma\right)$$