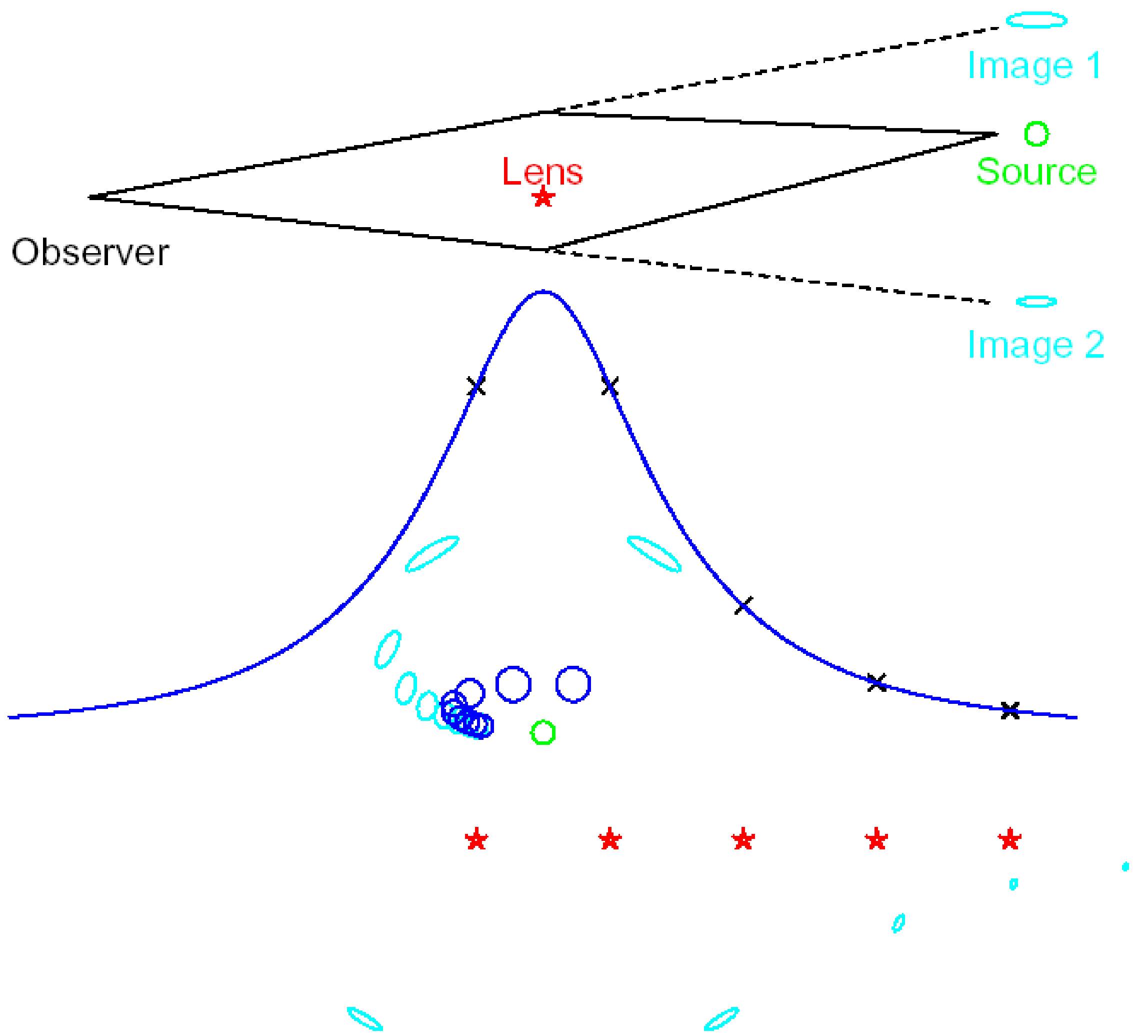
Microlensing I:
Point Lenses & Multiple Lenses

Andy Gould (Ohio State)



Generation 1

Liebes 1964, Phys Rev, 133, B835

Many practical examples, including planets

Refsdal 1964, MNRAS, 128, 259

Mass measurement of Isolated Star

Refsdal 1966, MNRAS, 134, 315

Space-Based Parallaxes

Paczynski 1986, ApJ, 304, 1

Proposed First Practical Experiment

Generation 0

Eddington 1920, Space, Time, and Gravitation

Chwolson 1924, Astron. Nachr. 221, 329

Einstein 1936a, Science, 84, 506

"Some time ago R.W. Mandl paid me a visit and asked me to publish the results of a little calculation, which I had made at his request .... there is no great chance of observing this phenomenon."

Einstein 1936b (private letter to Science editor)

"Let me also thank you for your cooperation with the little publication, which Mister Mandl squeezed out of me. It is of little value, but it makes the poor guy happy."

Generation -1: Einstein (1912)

[Renn, Sauer, Stachel 1997, Science 275, 184]

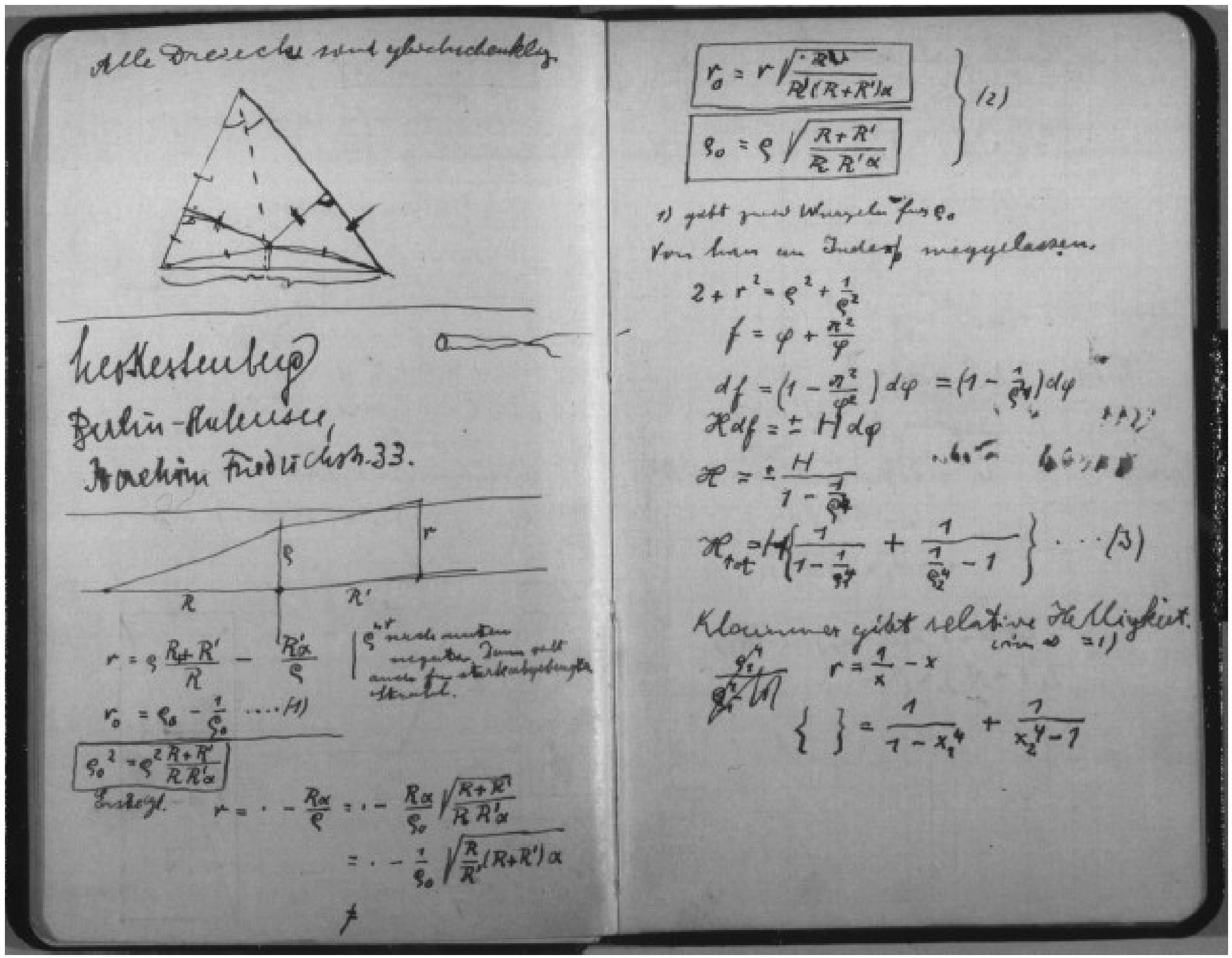
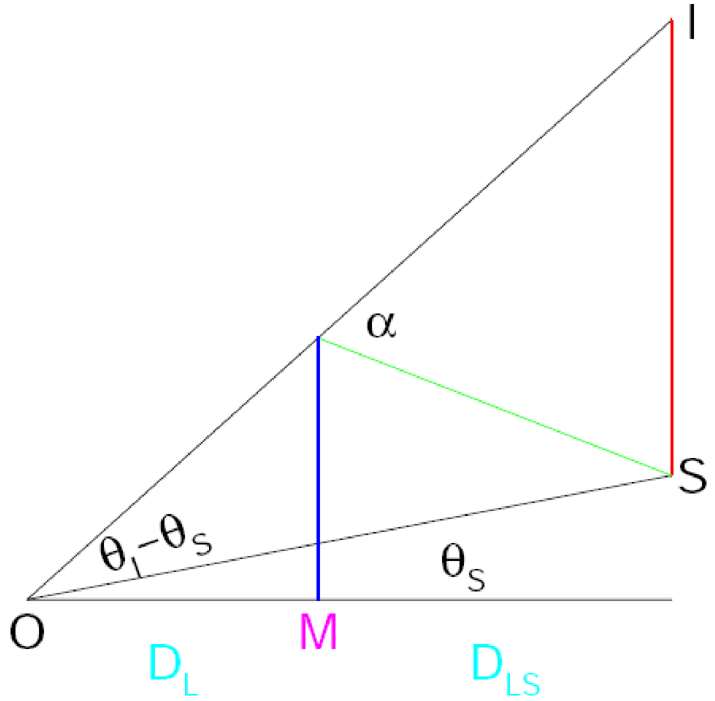


Fig. 1. Notes about gravitational lensing dated to 1912 on two pages of Einstein's scratch notebook (*12*). [Reproduced with permission of the Einstein Archives, Jewish National and University Library, Hebrew University of Jerusalem]

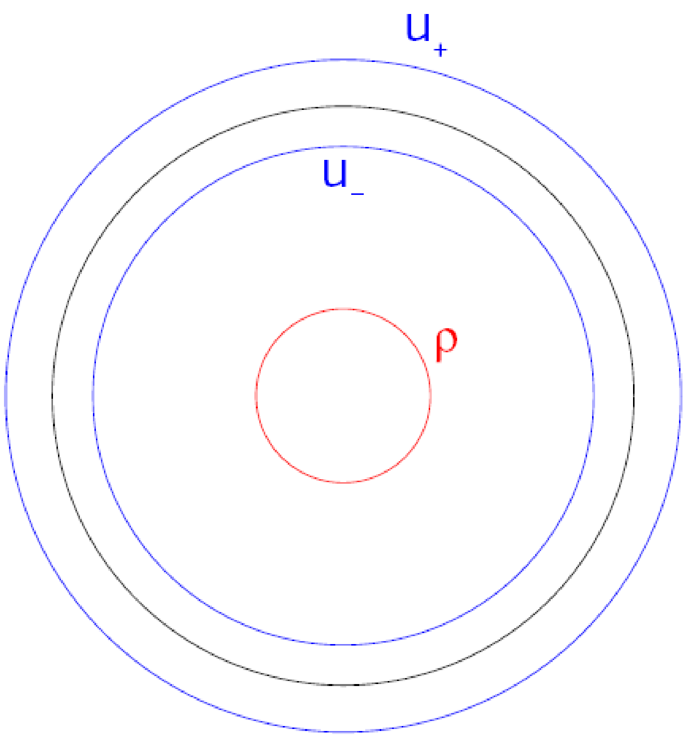


$$(\theta_I - \theta_S) D_S = \alpha D_{LS}$$

$$\alpha = 4GM/(D_L \theta_I c^2)$$

$$(\theta_I - \theta_S)\theta_I = \theta_E{}^2 = (4GM/c^2)(D_{LS}/D_L D_S)$$

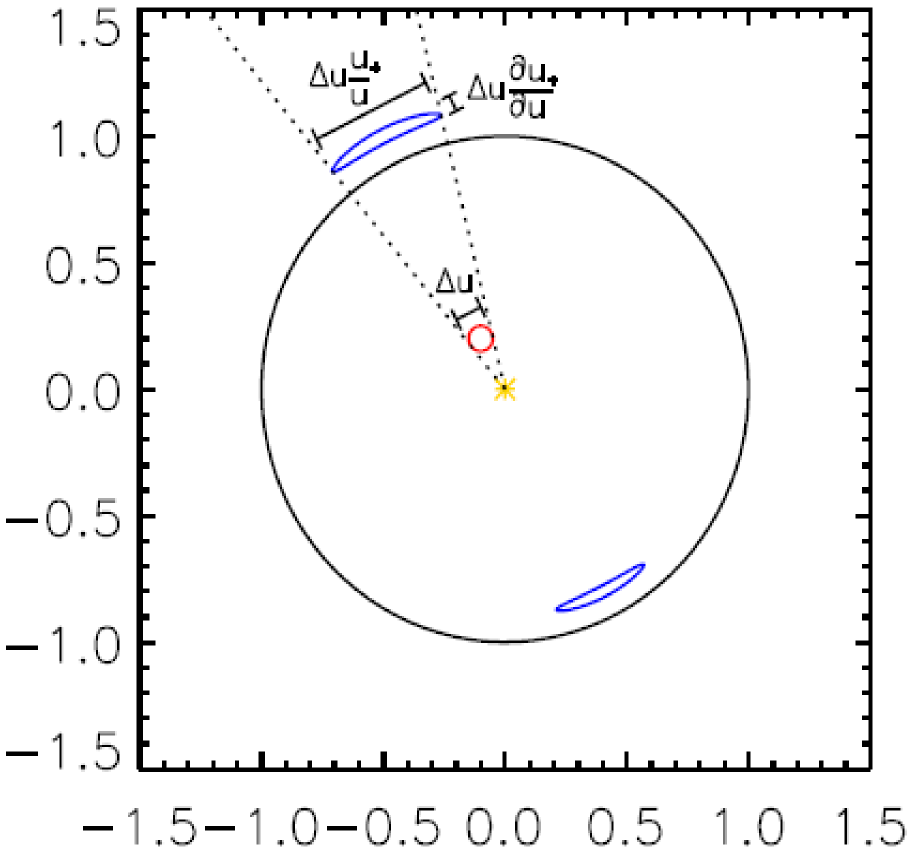
$$\theta_I/\theta_E = [u +/- (u^2 + 4)^{1/2}]/2; \;\; u=\theta_S/\theta_E$$



Point-Lens Magnification

$$A = \frac{u^2+2}{u\sqrt{u^2+4}}$$

$$\begin{aligned}
u_\pm &= \frac{\sqrt{u^2+4}}{2} \pm u \\
A_\pm &= \pm \frac{u_\pm}{u}\frac{\partial u_\pm}{\partial u} \\
&= \pm \frac{1}{2}\frac{\partial u_\pm^2}{\partial u^2} \\
A_+ - A_- &= \frac{1}{2}\left(\frac{\partial u_+^2}{\partial u^2} + \frac{\partial u_-^2}{\partial u^2}\right) = \frac{\partial(u^2+2)}{\partial u^2} = 1 \\
A_\pm &= \frac{A \pm 1}{2}
\end{aligned}$$

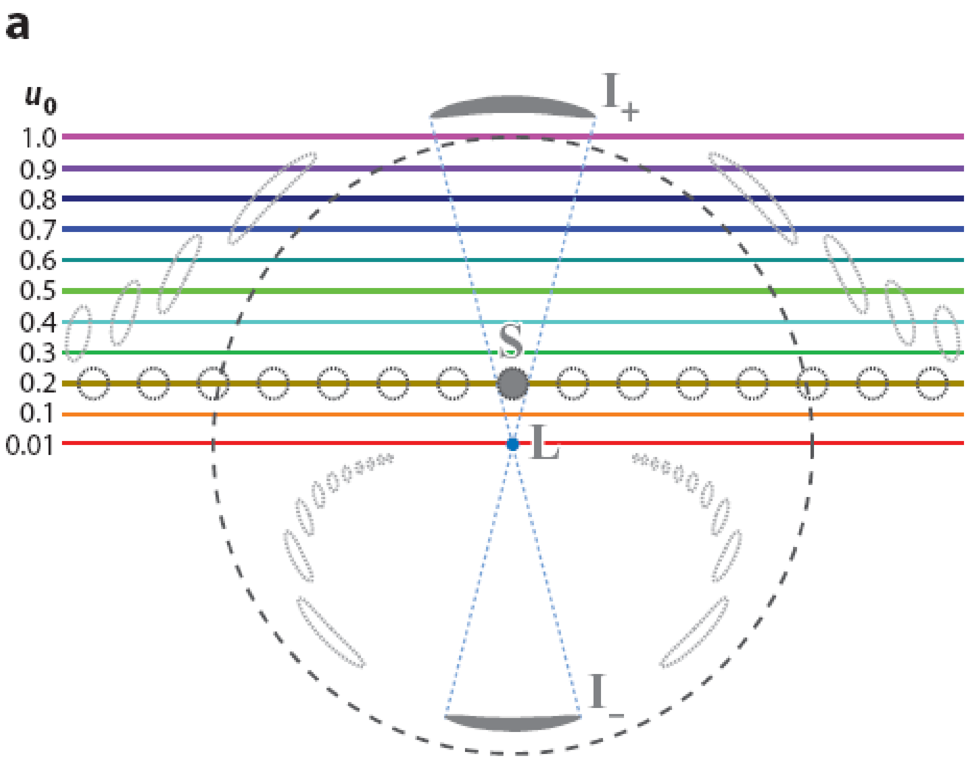


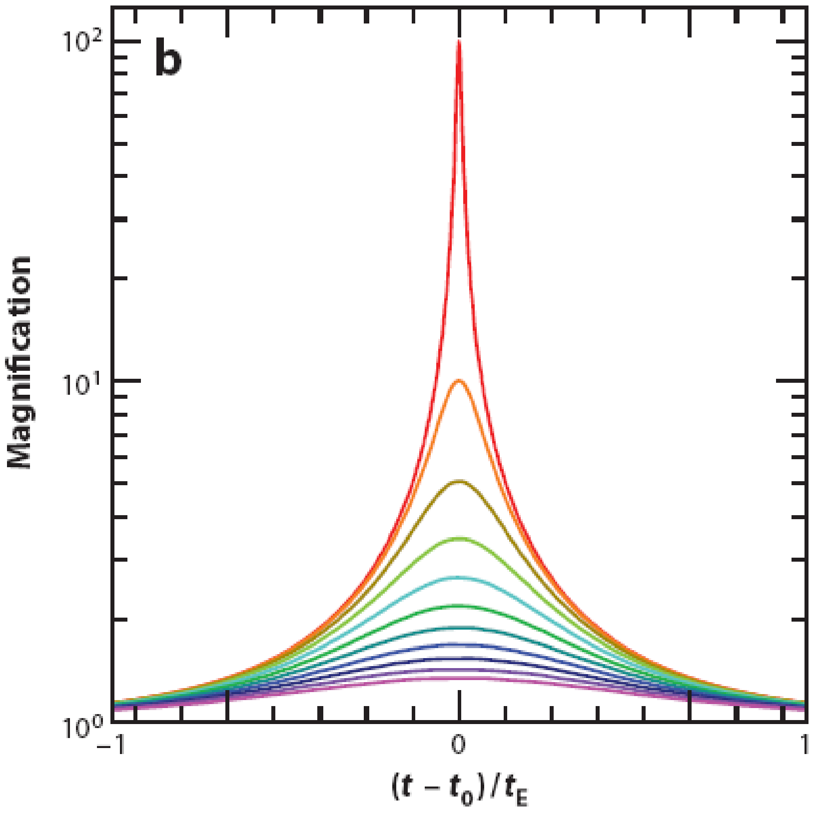
$$A = A_+ + A_- \quad = \quad \frac{1}{2}\left(\frac{\partial u_+^2}{\partial u^2} - \frac{\partial u_-^2}{\partial u^2}\right) = \frac{\partial(u\sqrt{u^2+4})}{2u\partial u} = \frac{u^2+2}{u\sqrt{u^2+4}}$$

Point-Lens Limiting Formulae

$$
\begin{aligned}
A(u) &= \frac{1}{u}\frac{1+u^2/2}{\sqrt{1+u^2/4}} \rightarrow \frac{1}{u}\left(1+\frac{3}{8}u^2\right) \qquad (u \ll 1) \\
A(u) &= \left(1-\frac{4}{(u^2+2)^2}\right)^{-1/2} \rightarrow 1+\frac{2}{(u^2+2)^2} \qquad (u \gg 1) \\
A(1) &= \frac{3}{\sqrt{5}} \simeq 1.34 \\
u(A) &= \sqrt{2[(1-A^{-2})^{-1/2}-1]}
\end{aligned}
$$

Point-Lens Light Curves





3 Features & 3 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E

Finite Source "Attenuation"

$$A(u) \to \frac{1}{u}$$

$$A_{\rm fin}(u|\rho) = \frac{1}{\pi \rho^2} \int_0^{2\pi} d\theta \int_0^{\rho} d\bar{\rho}\, \rho A[(u + \bar{\rho}\cos\theta)^2 + (\bar{\rho}\sin\theta)^2]$$

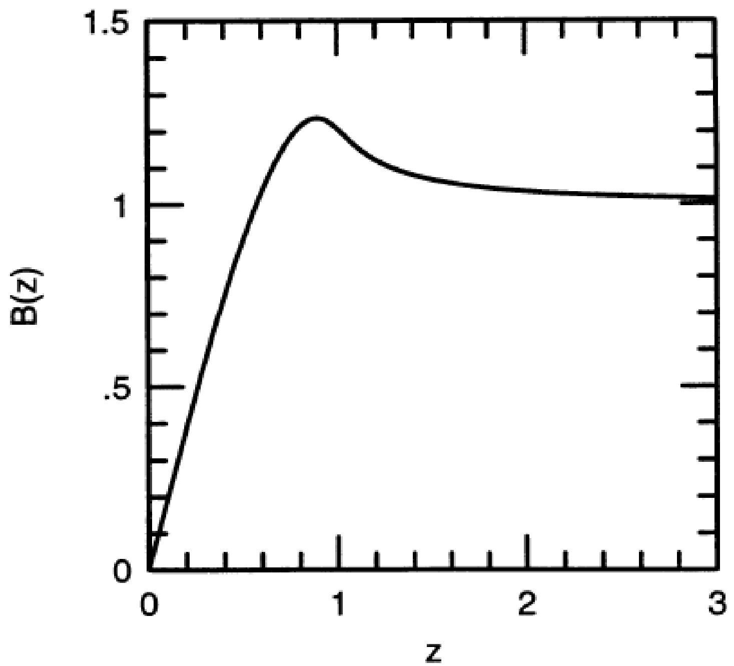
$$A_{\rm fin}(u|\rho) = \frac{B(z)}{u}; \qquad z \equiv \frac{u}{\rho}$$

$$B(z) = \frac{z^2}{\pi} \int_0^{2\pi} d\theta \int_0^{1/z} dq\, q(1 + q^2 + 2q\cos\theta)^{-1/2}$$

$$A_{\rm fin}(u|\rho) = \frac{B(z)}{u} \to A(u)B(z)$$

Finite Source “Attenuation”

$$A_{\text{fin}}(u|\rho) = \frac{B(z)}{u} \rightarrow A(u)B(z)$$

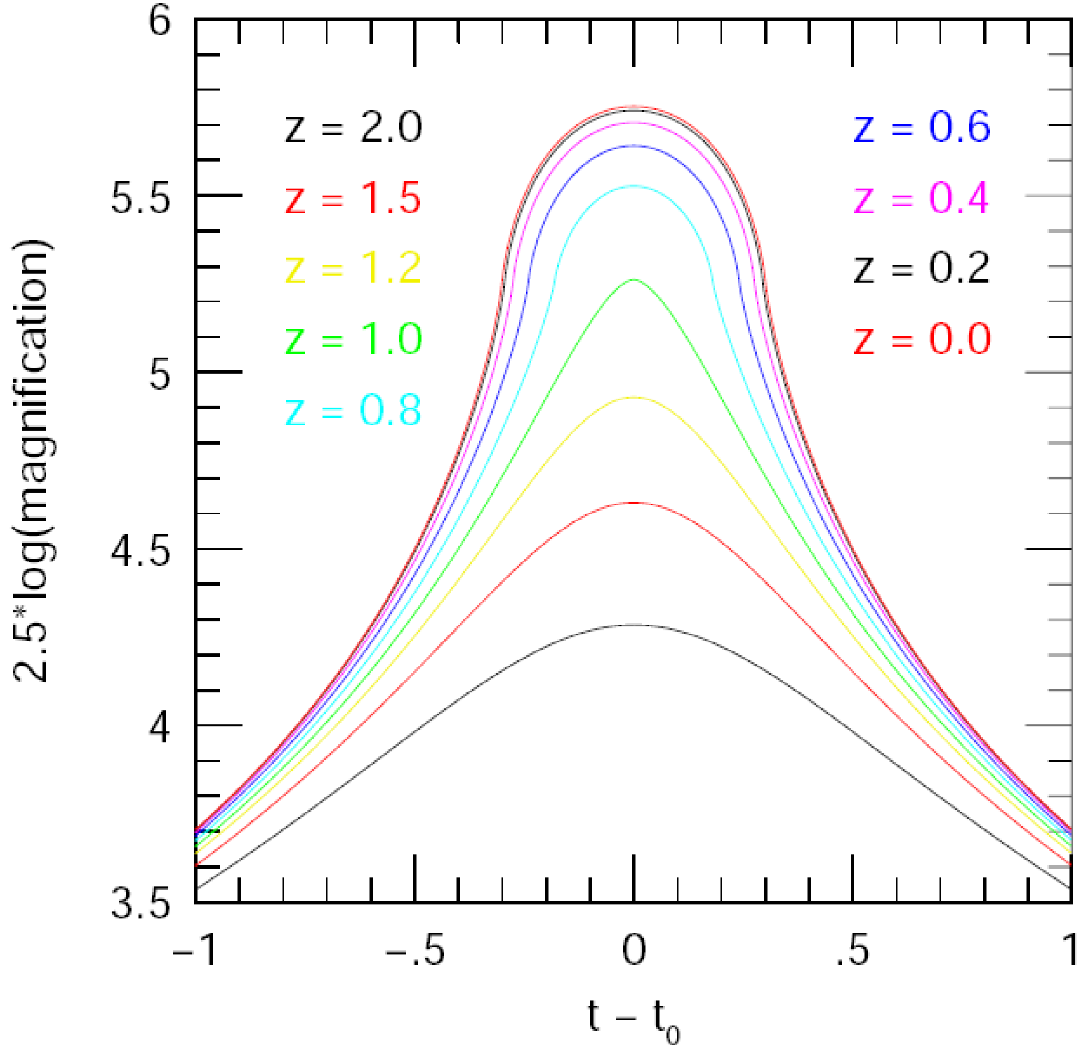


PROPER MOTIONS OF MACHOs

ANDREW GOULD

THE ASTROPHYSICAL JOURNAL, 421:L71–L74, 1994 February 1

Finite Source "Attenuation"

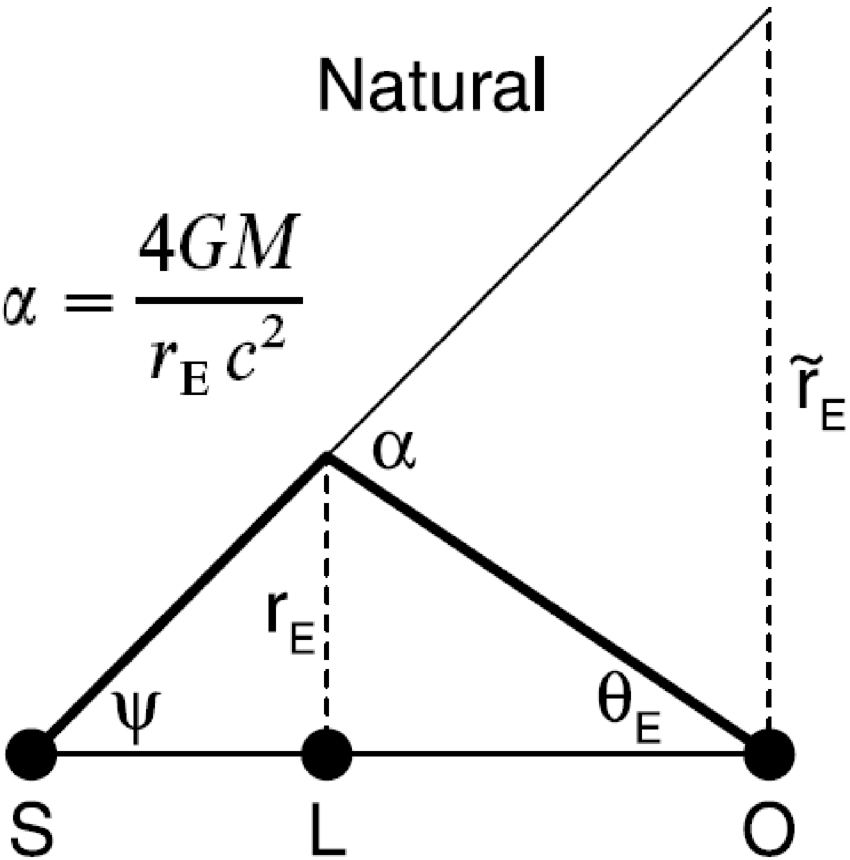


4 Features & 4 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Flattening of Peak	$t_* = \rho * t_E$

Relation of Mass and Distance to Lensing Observables

$$\alpha = \frac{4GM}{r_E c^2}$$

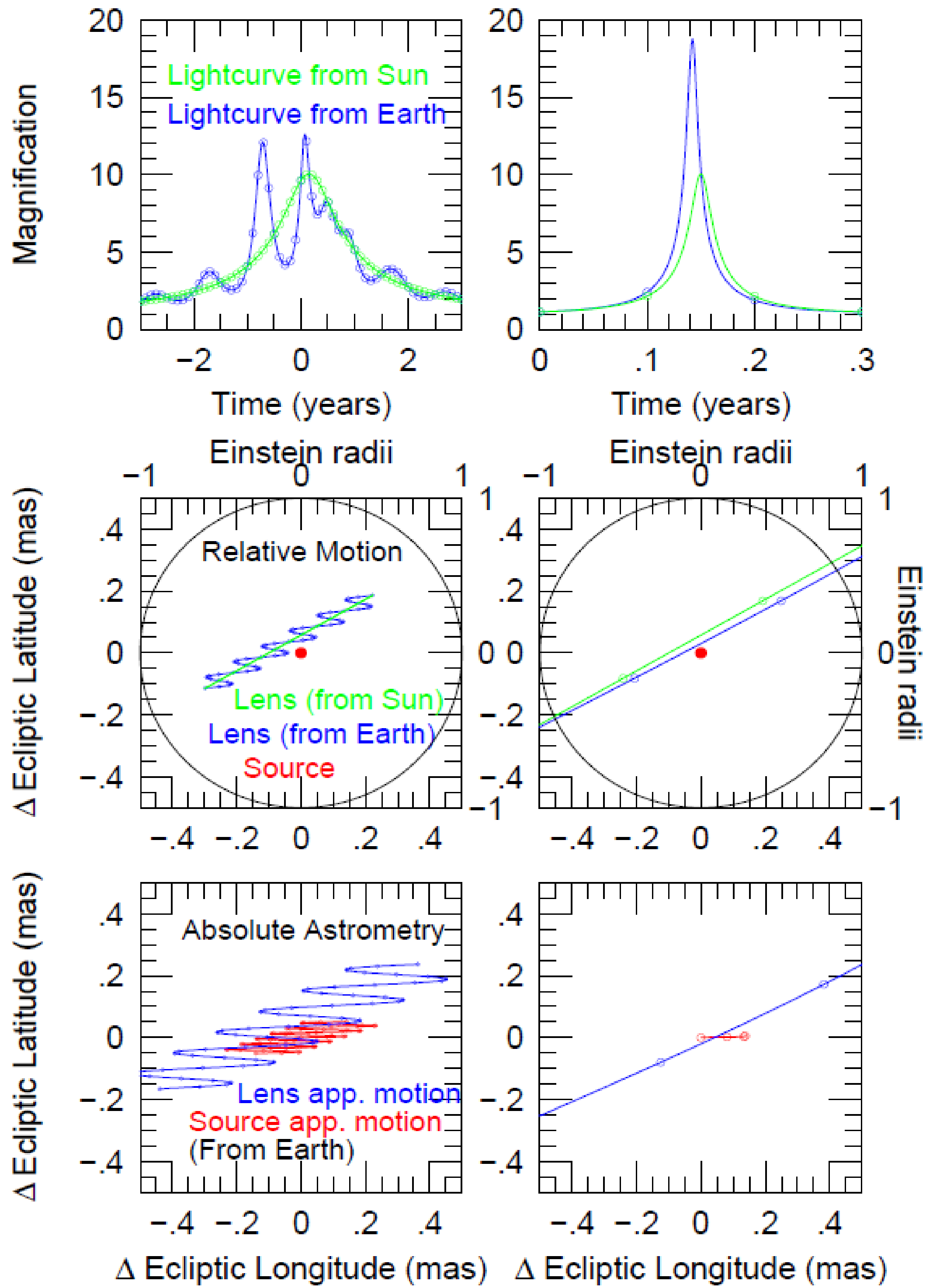


$$\alpha / \tilde{r}_E = \theta_E / r_E$$

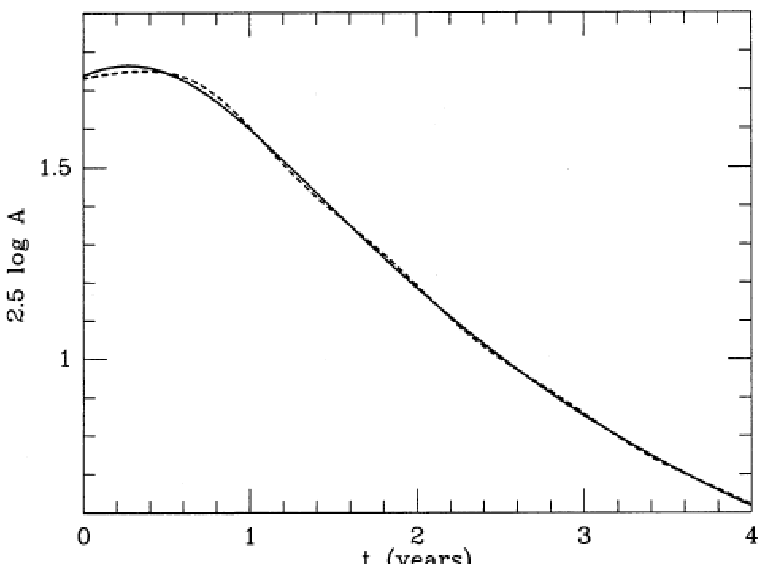
$$\theta_E \tilde{r}_E = \alpha r_E = \frac{4GM}{c^2}$$

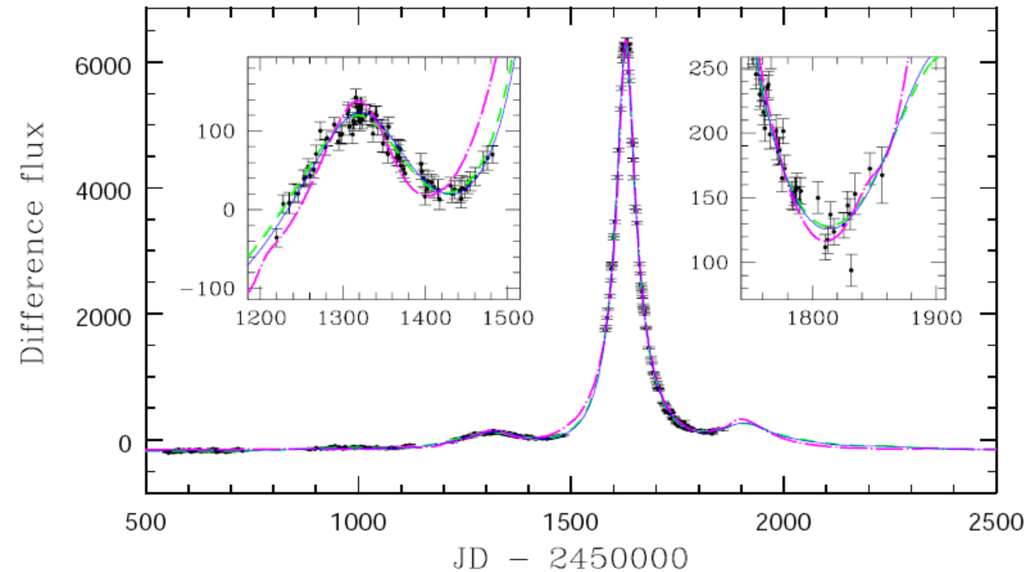
$$\theta_E = \alpha - \psi = \frac{\tilde{r}_E}{D_l} - \frac{\tilde{r}_E}{D_s} = \frac{\tilde{r}_E}{D_{rel}}$$

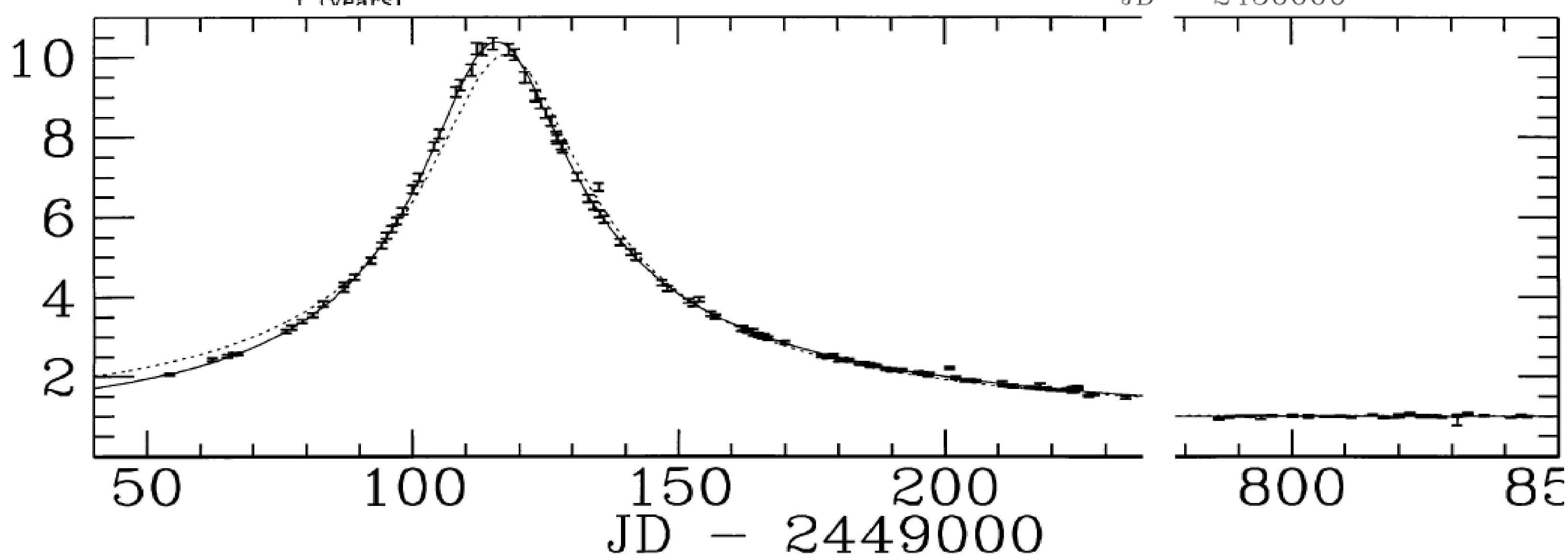
$$\tilde{r}_E = \sqrt{\frac{4GMD_{rel}}{c^2}} \qquad \theta_E = \sqrt{\frac{4GM}{D_{rel} c^2}}$$


Magnification
Lightcurve from Sun
Lightcurve from Earth
Time (years)
Time (years)
Einstein radii
Einstein radii
Relative Motion
Lens (from Sun)
Lens (from Earth)
Source
Δ Ecliptic Latitude (mas)
Einstein radii
Absolute Astrometry
Lens app. motion
Source app. motion
(From Earth)
Δ Ecliptic Latitude (mas)
Δ Ecliptic Longitude (mas)
Δ Ecliptic Longitude (mas)

To measure parallax:
Standard Observer-Plane Rulers



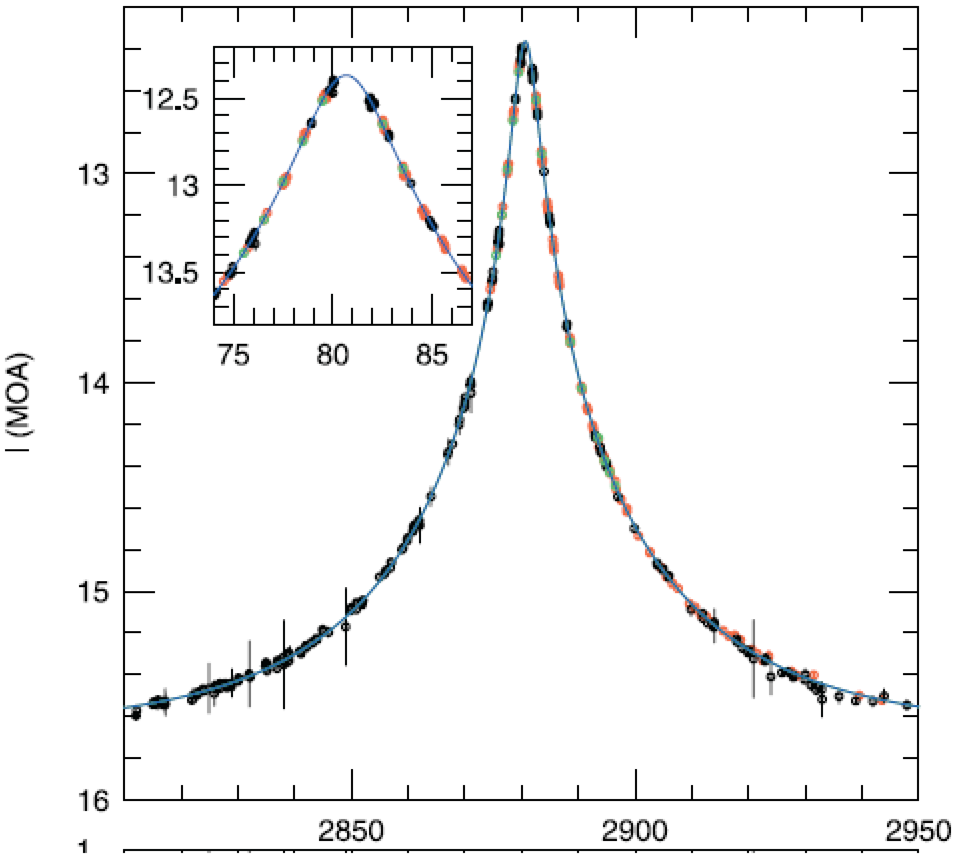




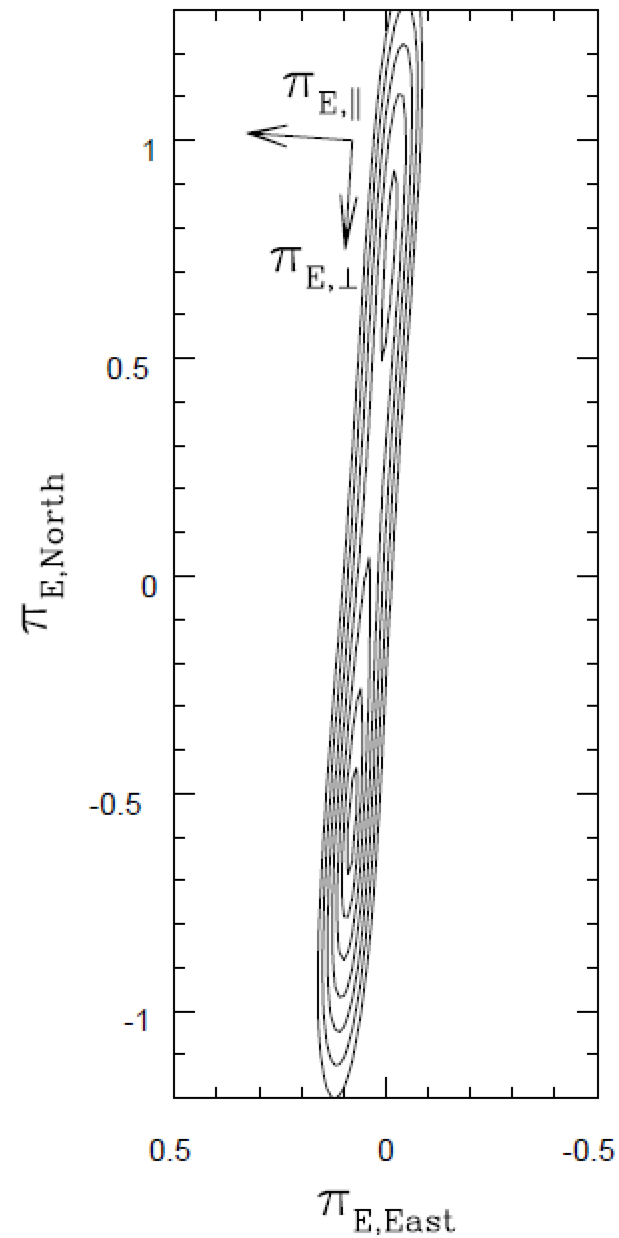
5 Features & 5 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Anti-symmetric Distort.	$\pi_{E,\mathrm{parallel}}$
Symmetric Distortion	$\pi_{E,\mathrm{perp}}$

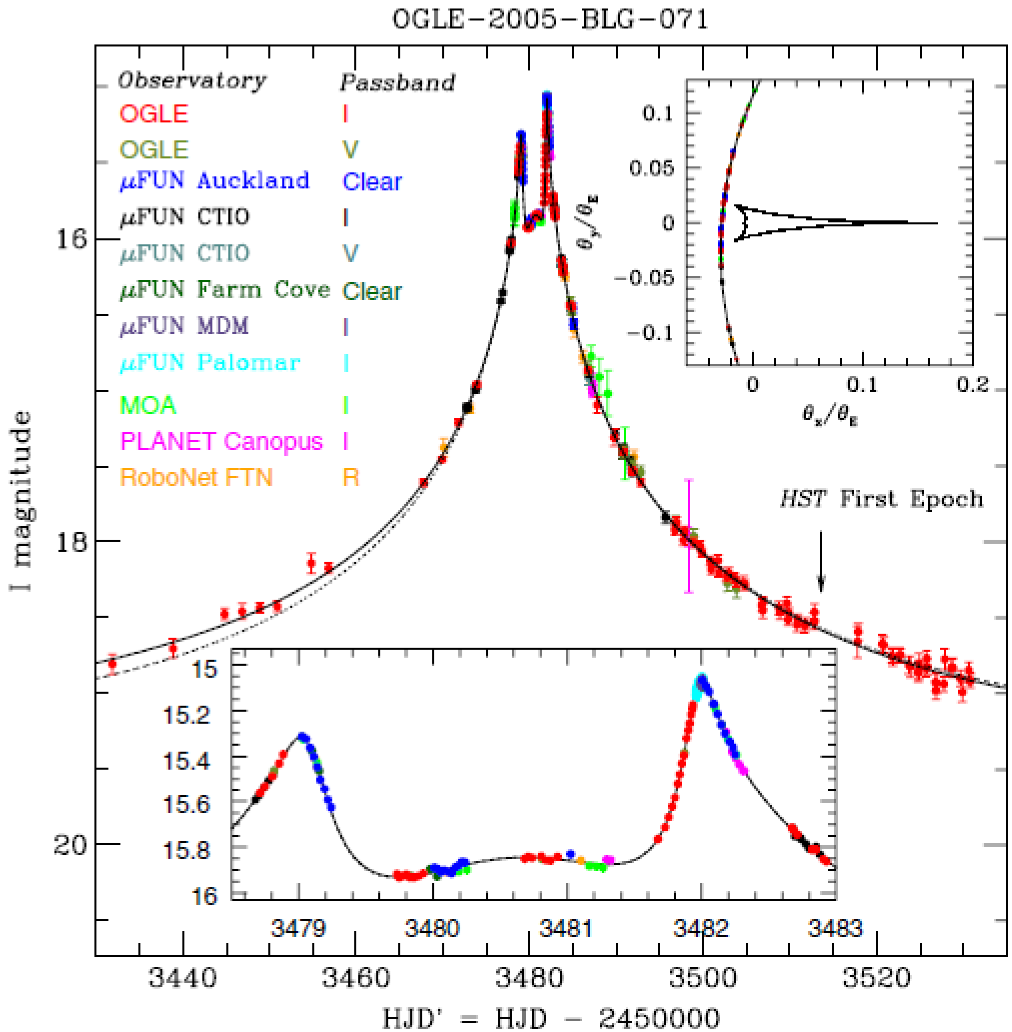
1-D Parallaxes Are "Common"

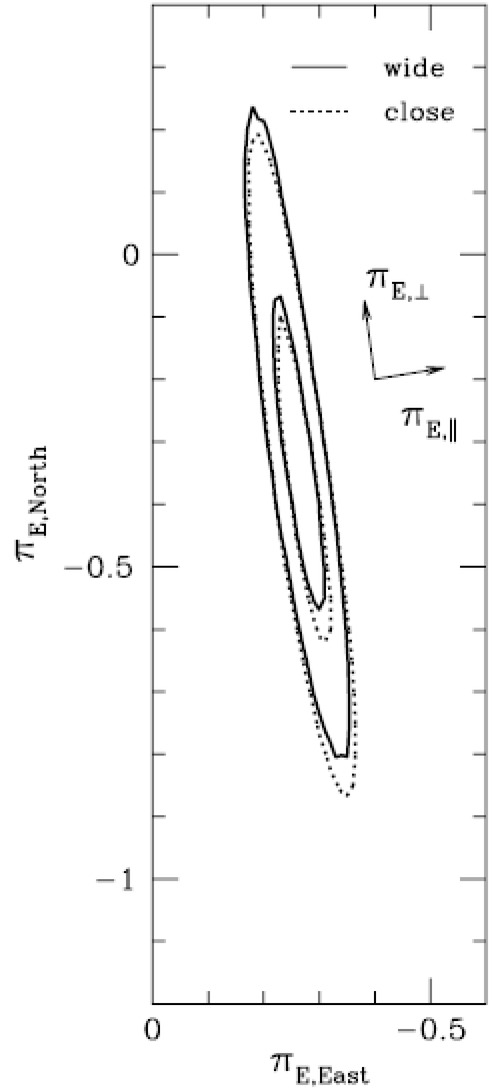


MOA-2003-BLG-37
Park et al. 2004, ApJ. 609, 166



1-D Parallaxes Are “Common”

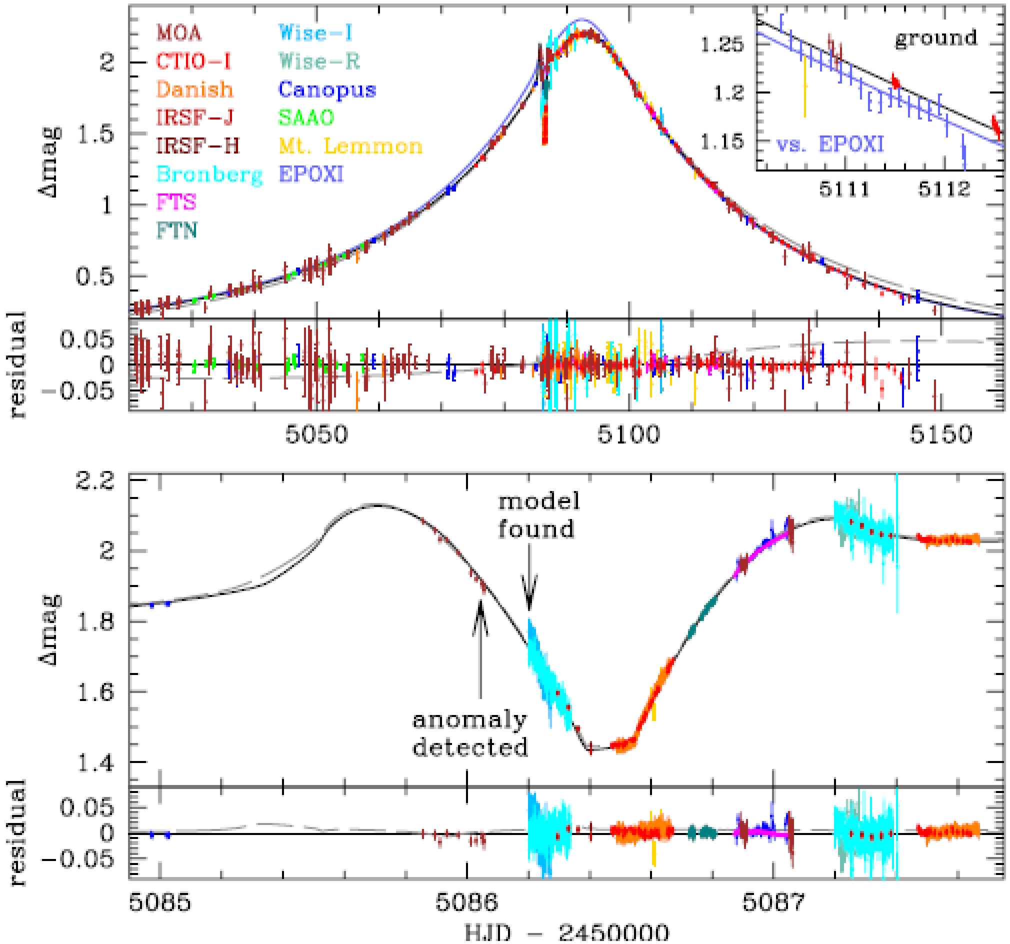


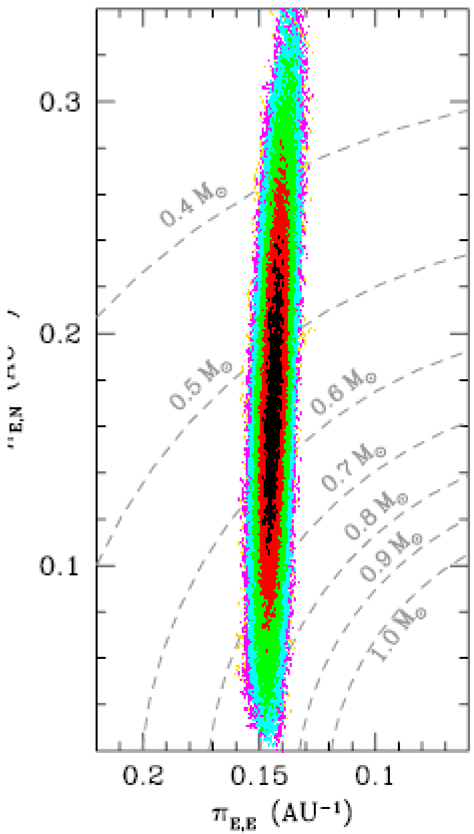


OGLE-2005-BLG-071

Dong et al. 2009, ApJ. 695, 970

1-D Parallaxes Are "Common"





MOA-2009-BLG-266

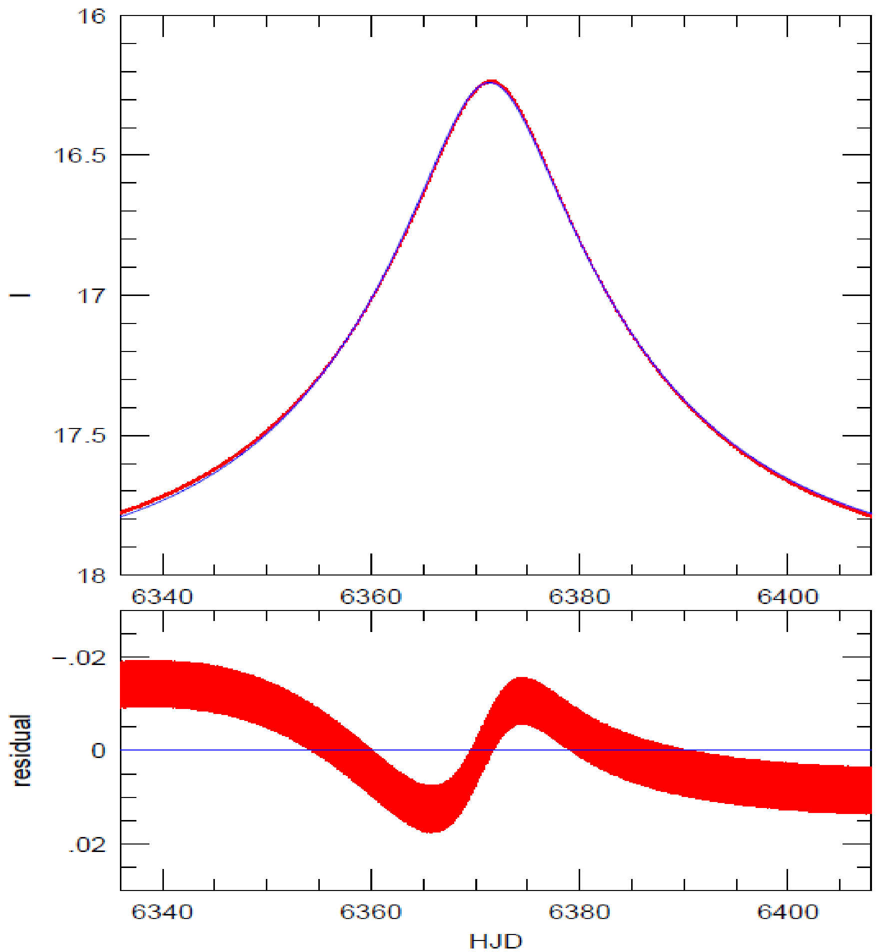
Muraki et al. 2011, ApJ, 741, 22

5 Features & 5 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Anti-symmetric Distort.	$\pi_{E,\parallel}$
Symmetric Distortion (tough!)	$\pi_{E,\perp}$

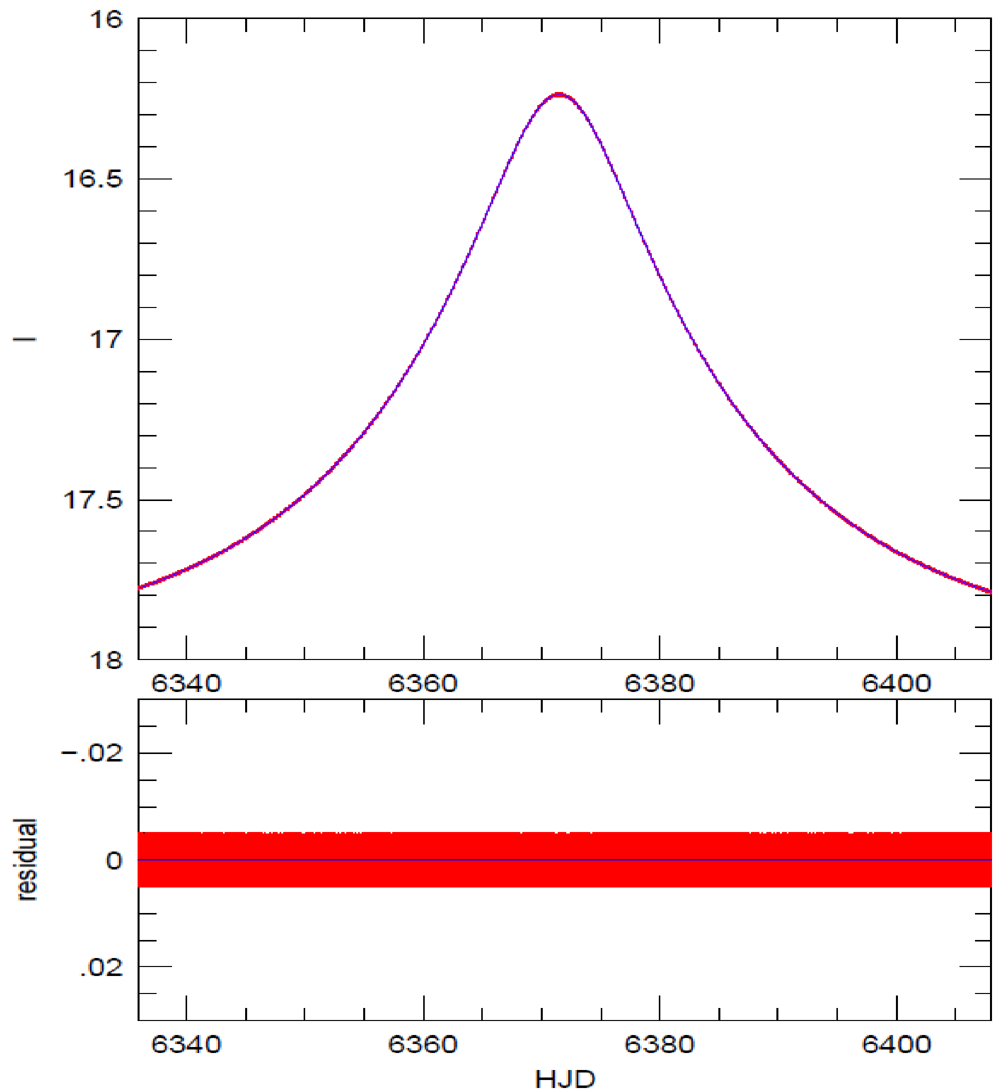
$\pi_{E,parallel}$ (square peg: round hole)

Component of π_E toward Sun

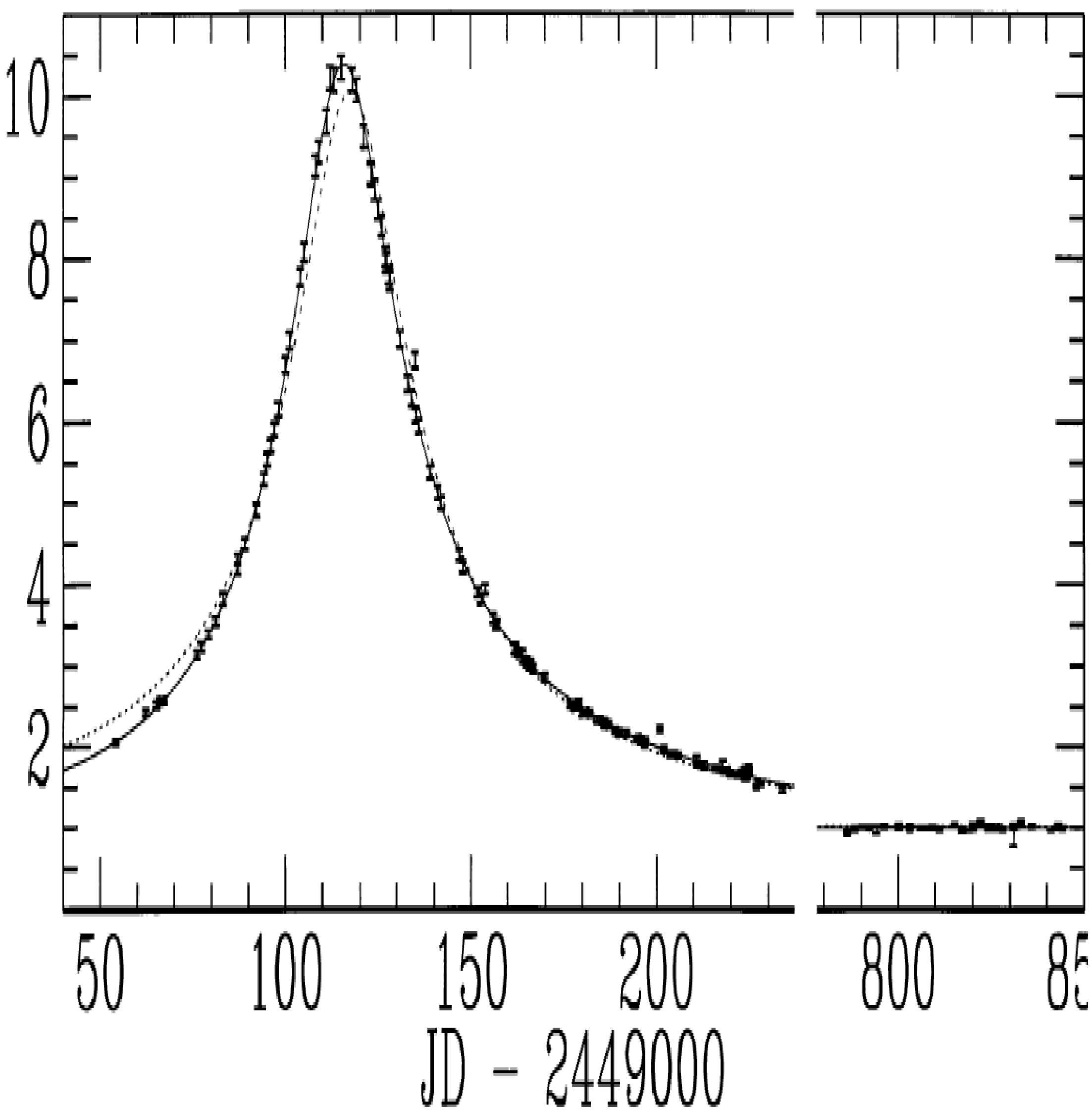


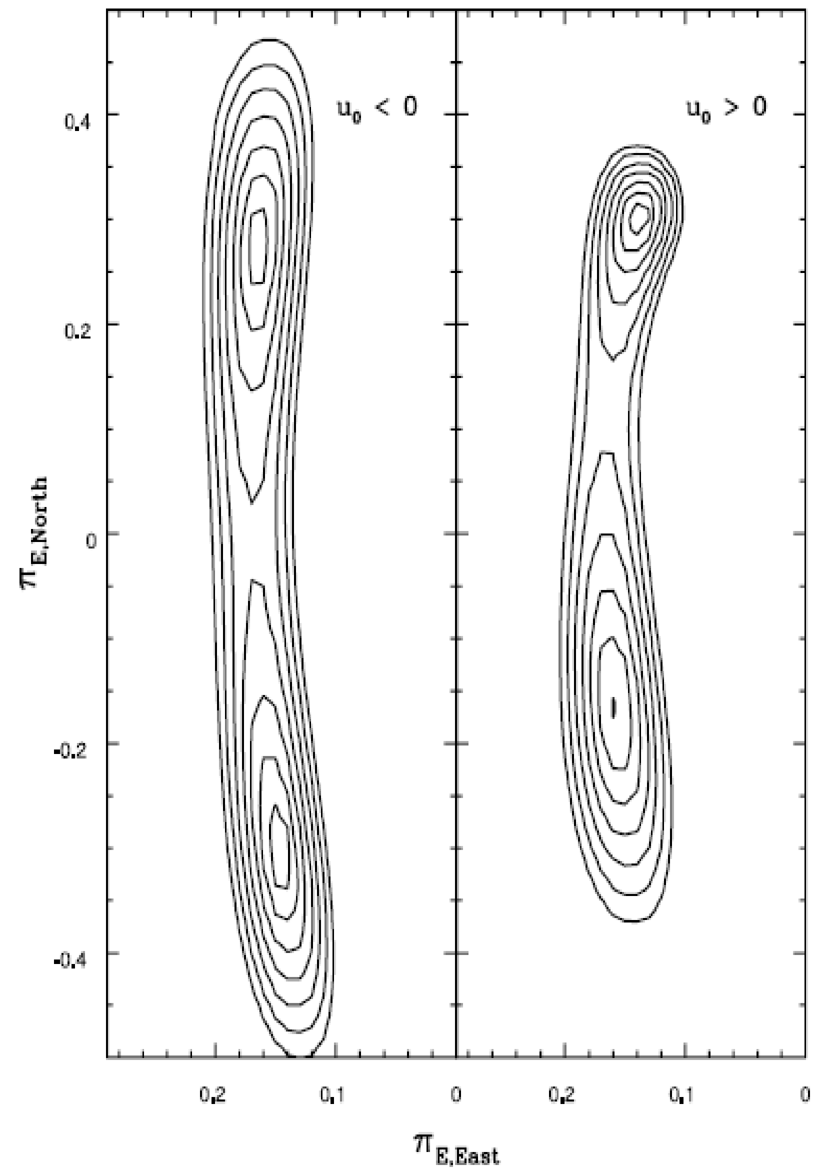
$\pi_{E,perp}$ (round peg: round hole)

Component of π_E perp to Sun



Parallax Degeneracies: Macho 104-C





Introducing Jerk-Parallax:

$$\pi_j \equiv \frac{4}{3}\frac{\boldsymbol{j}}{\alpha^2 t_{\rm E}}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{\omega} t + \pi_{\rm E}\left(\frac{1}{2}\boldsymbol{\alpha} t^2 + \frac{1}{6}\boldsymbol{j} t^3 + \ldots\right)$$

$$u^2 = \sum_{i=0}^{\infty} C_i t^i \qquad C_0 = u_0^2, \quad C_1 = 0,$$

$$C_2 = -\alpha\, u_0 \pi_{{\rm E},\perp} + t_{\rm E}^{-2}$$

$$C_3 = \alpha\frac{\pi_{{\rm E},\parallel}}{t_{\rm E}} + \frac{1}{4}\alpha^2 t_{\rm E} u_0 \boldsymbol{\pi}_{\rm E} \times \boldsymbol{\pi}_j$$

$$C_4 = \frac{\alpha^2}{4}(\pi_{\rm E}^2 + \boldsymbol{\pi}_j \cdot \boldsymbol{\pi}_{\rm E}) + \frac{1}{12}\frac{\Omega_\oplus^2}{\alpha} u_0 \pi_{{\rm E},\perp}$$

Theory vs. Observation

$$u_0 = 0 \Rightarrow C_0 = C_1 = 0; \qquad C_2 = t_E^{-2}$$

$$C_3 = \frac{\alpha}{t_E}\pi_{E,\parallel}; \qquad C_4 = \frac{\alpha^2}{4}\vec{\pi}_E \cdot (\vec{\pi}_E + \vec{\pi}_j)$$

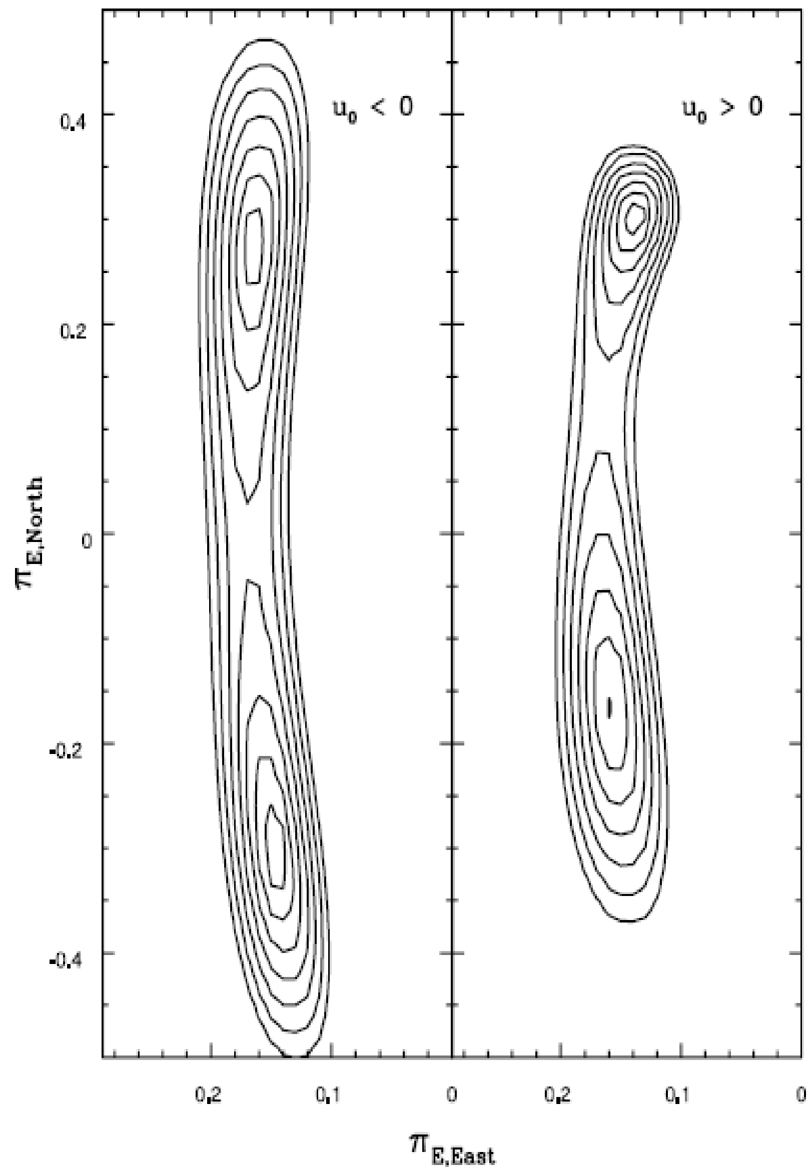
$$t'_E = t_E; \qquad \pi_{E,\parallel} = \pi'_{E,\parallel}$$

$$\vec{\pi}'_E \cdot (\vec{\pi}'_E + \vec{\pi}_j) = \vec{\pi}_E \cdot (\vec{\pi}_E + \vec{\pi}_j)$$

$$\pi'_{E,\perp}(\pi'_{E,\perp} + \pi_{j,\perp}) = \pi_{E,\perp}(\pi_{E,\perp} + \pi_{j,\perp})$$

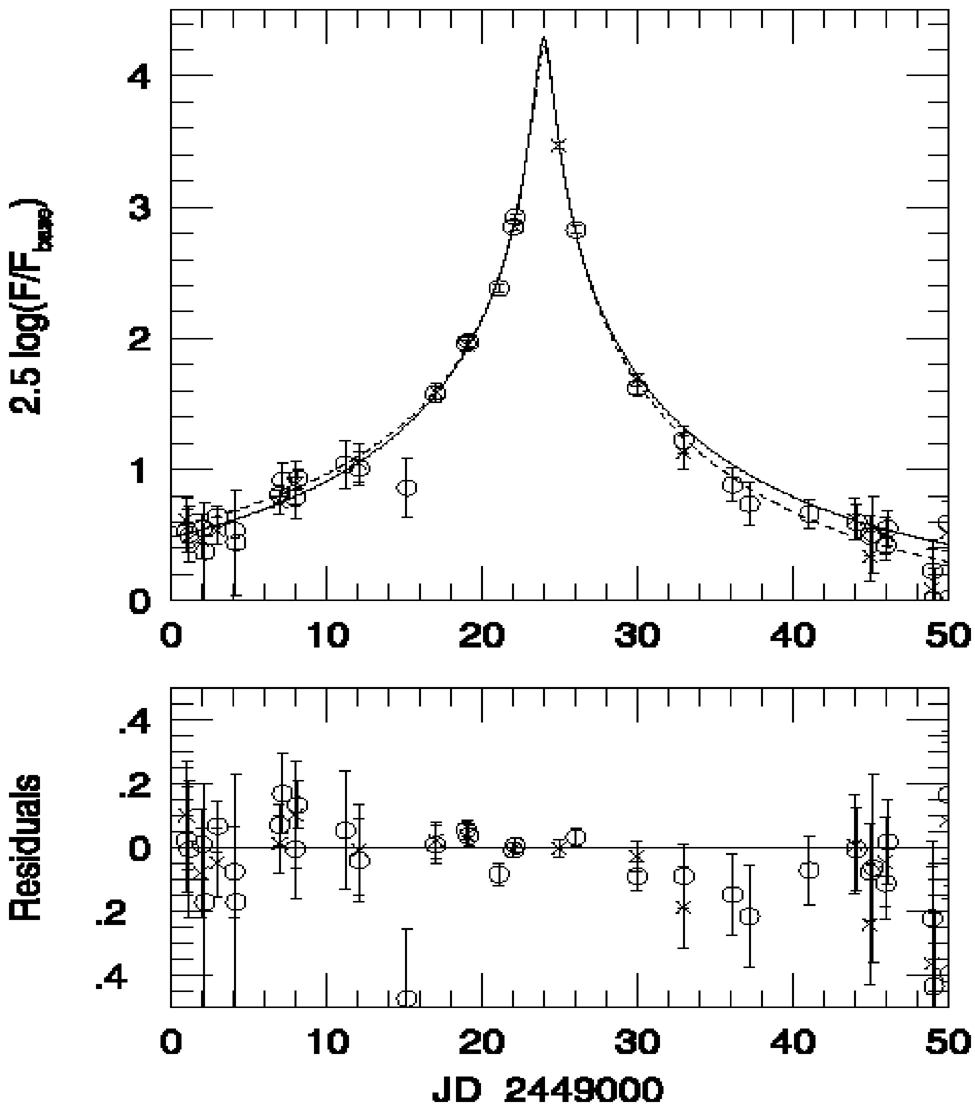
$$\pi'_{E,\perp} = -(\pi_{E,\perp} + \pi_{j,\perp})$$

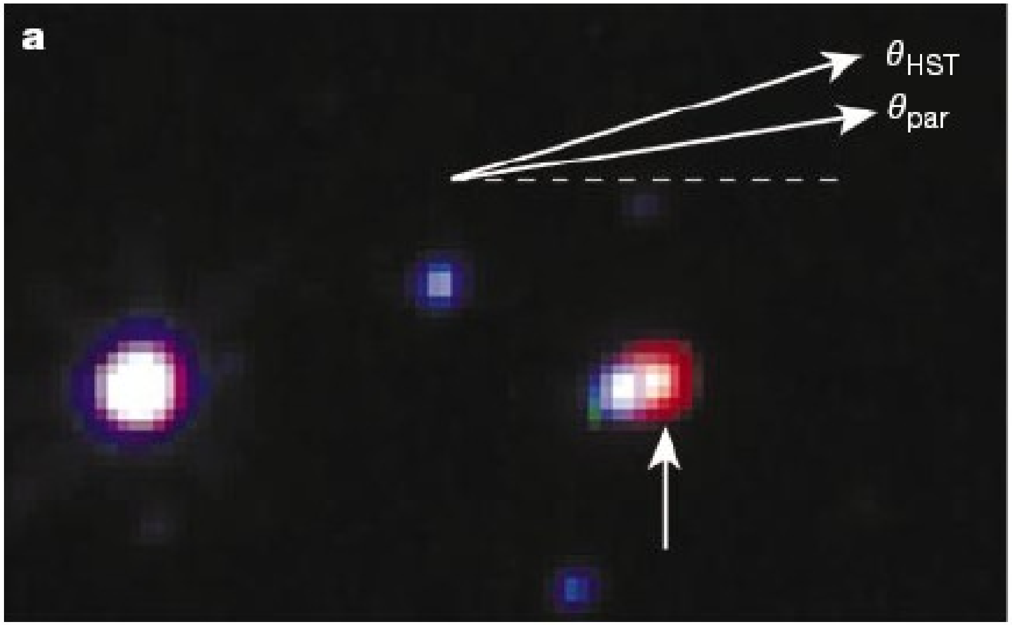
$$\pi_{j,\perp} \to -\frac{4}{3}\,\frac{\mathrm{yr}}{2\pi t_E}\,\frac{\sin\beta}{(\cos^2\psi\sin^2\beta + \sin^2\psi)^{3/2}}$$



MACHO-LMC-5

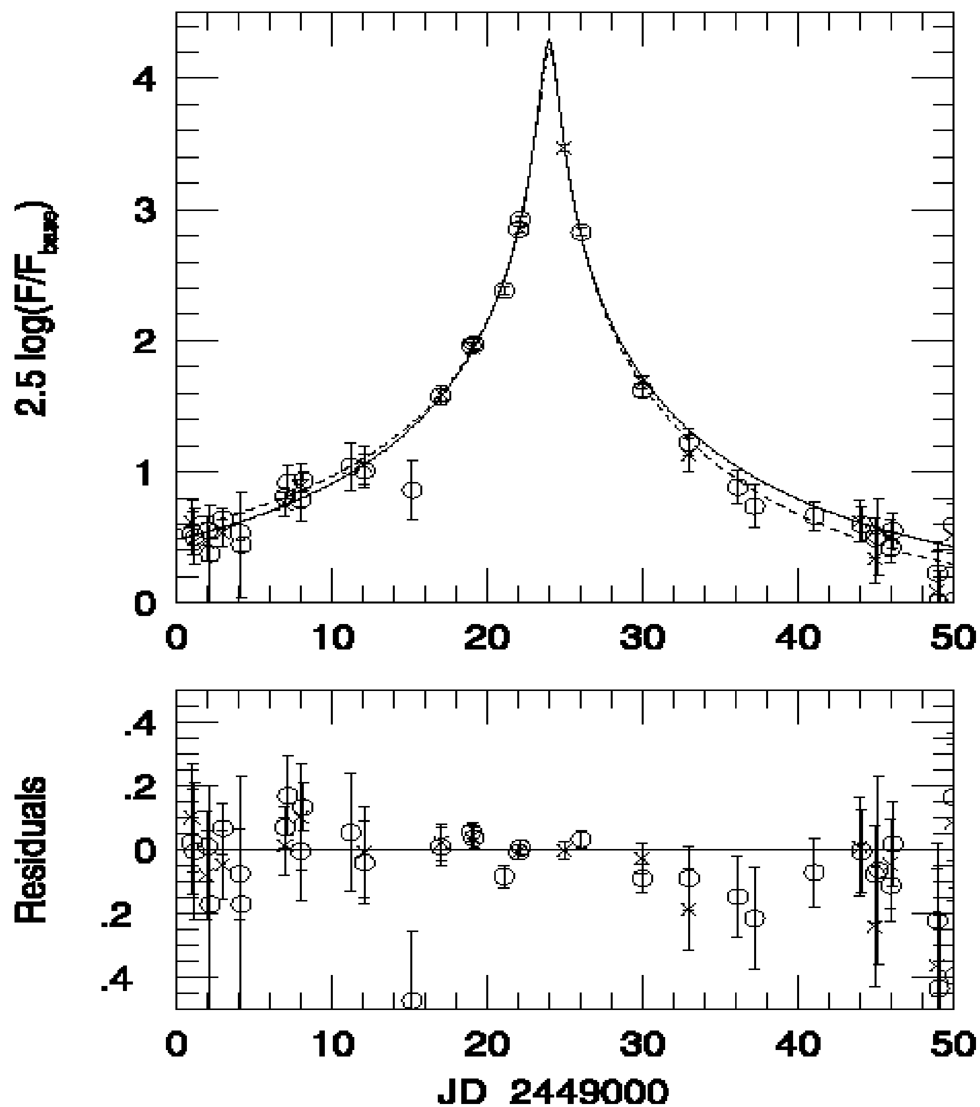
Angular Einstein radius from proper motion





MACHO-LMC-5

First Jerk-Parallax Degeneracy



Macho LMC-5:
Analytic formula works

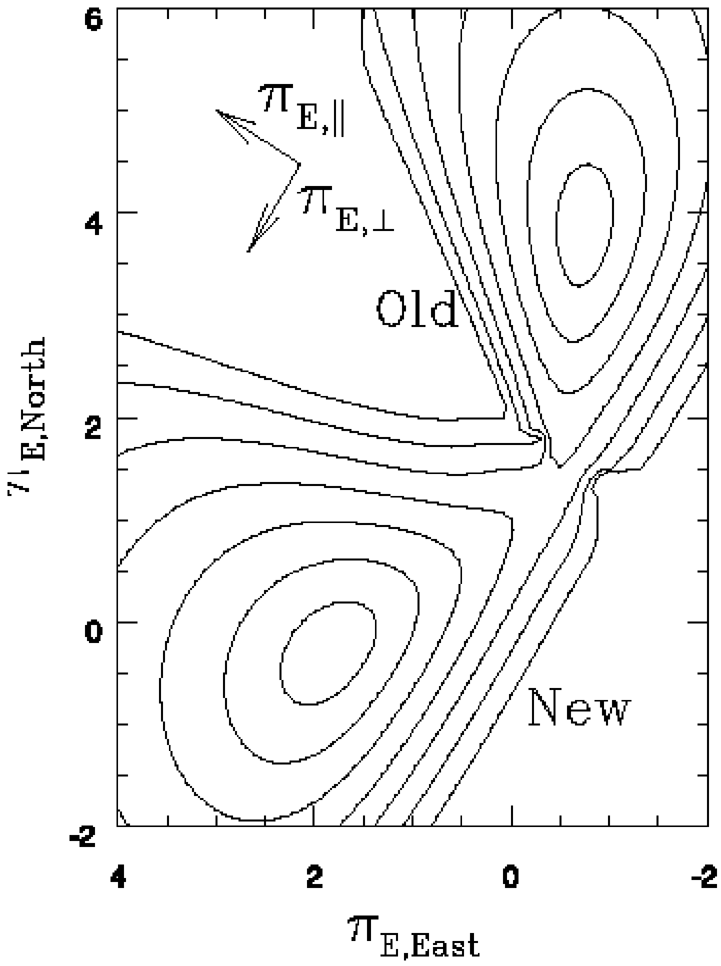
$$\pi_{j,\perp} \to -\frac{4}{3}\,\frac{\mathrm{yr}}{2\pi t_{\mathrm{E}}}\,\frac{\sin\beta}{(\cos^2\psi\sin^2\beta+\sin^2\psi)^{3/2}}$$

$$\mathrm{LMC} \Rightarrow \beta\sim -90^\circ \Rightarrow \pi_{j,\perp} \to \frac{78\,\mathrm{day}}{t_{\mathrm{E}}} \to 2.4$$

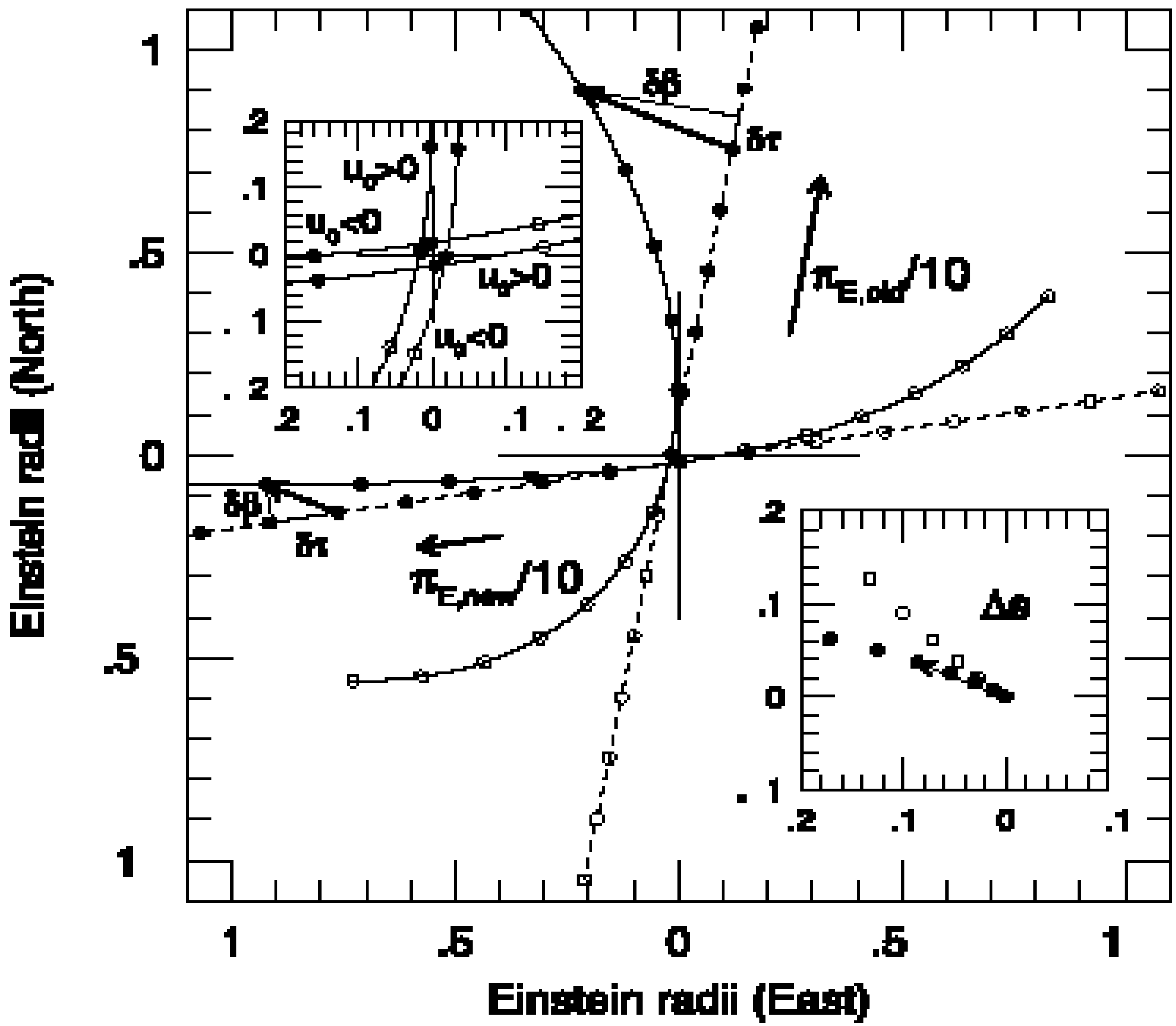
$$\pi_{\mathrm{E},\perp,\mathrm{old}} = -3.7$$

$$\Rightarrow \pi_{\mathrm{E},\perp,\mathrm{new}} = -(\pi_{\mathrm{E},\perp,\mathrm{old}} + \pi_{j,\perp})$$

$$= -(-3.7+2.4) = +1.3$$



Implies that very different trajectories can generate the same lightcurve.

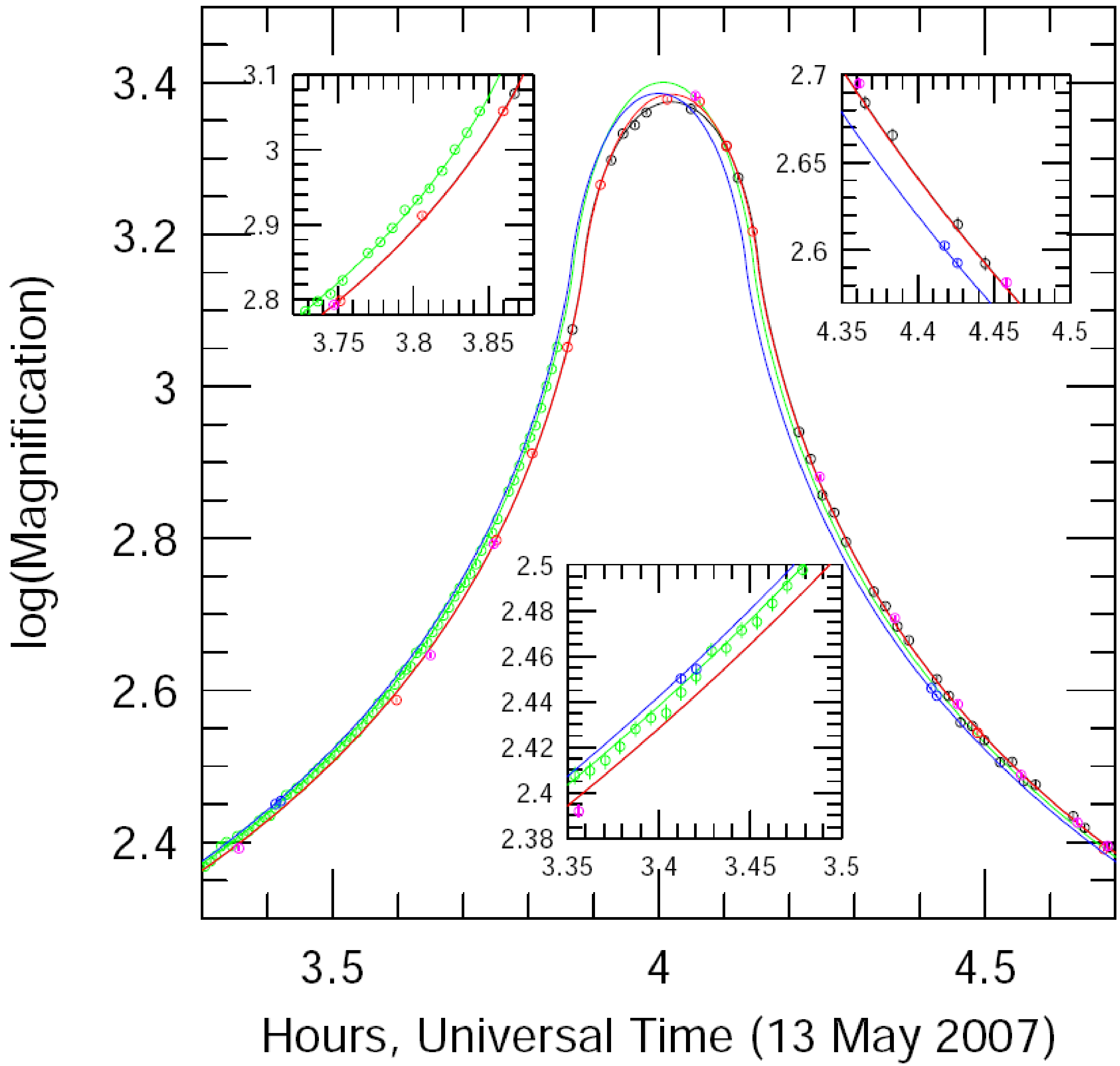


6 Features & 6 Parameters

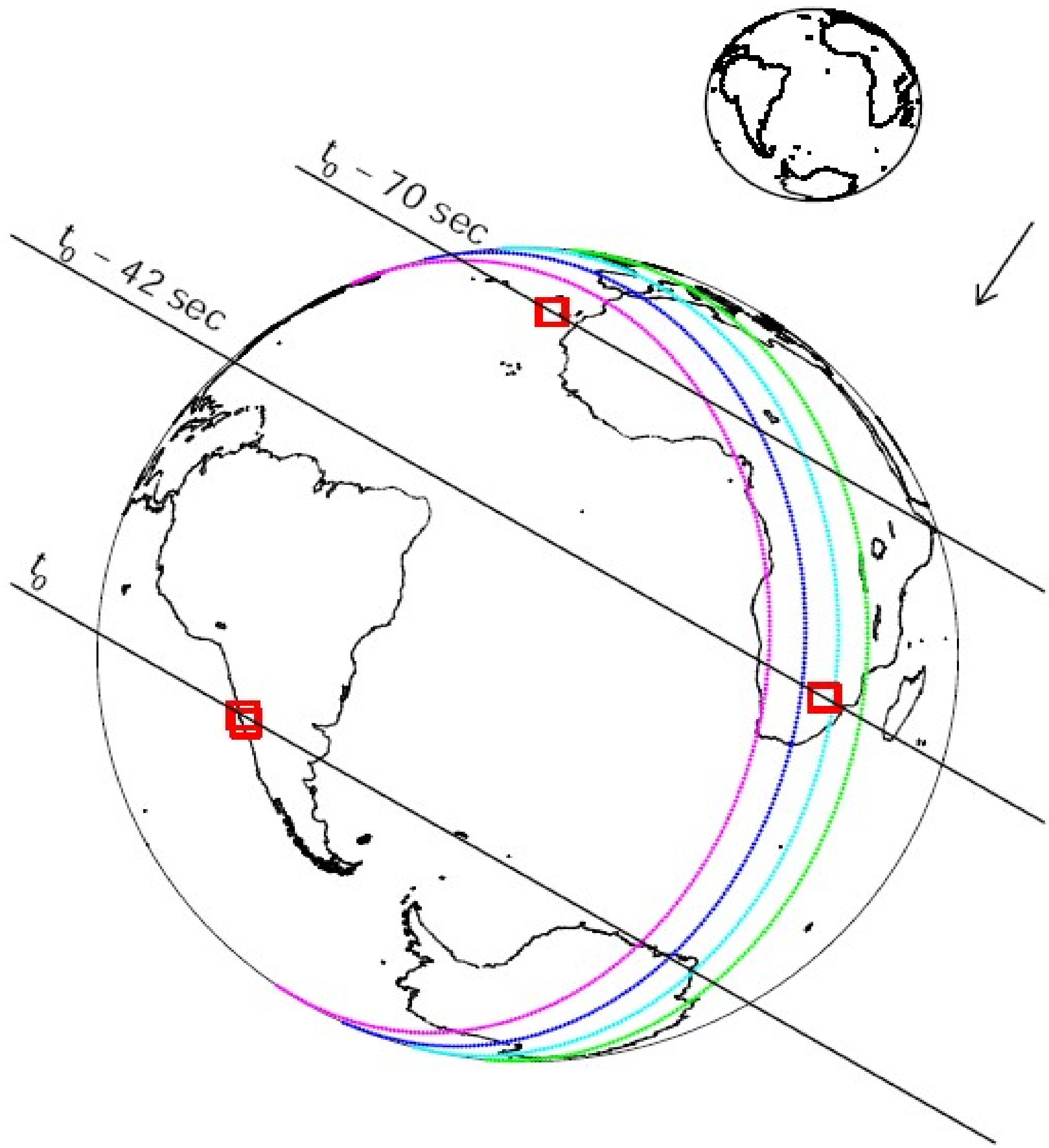
Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Anti-symmetric Distort.	$\pi_{E,\mathrm{parallel}}$
Symmetric Distortion	$\pi_{E,\mathrm{perp}}$
Flattening of Peak	$t_* = \rho * t_E$

OGLE-2007-BLG-224

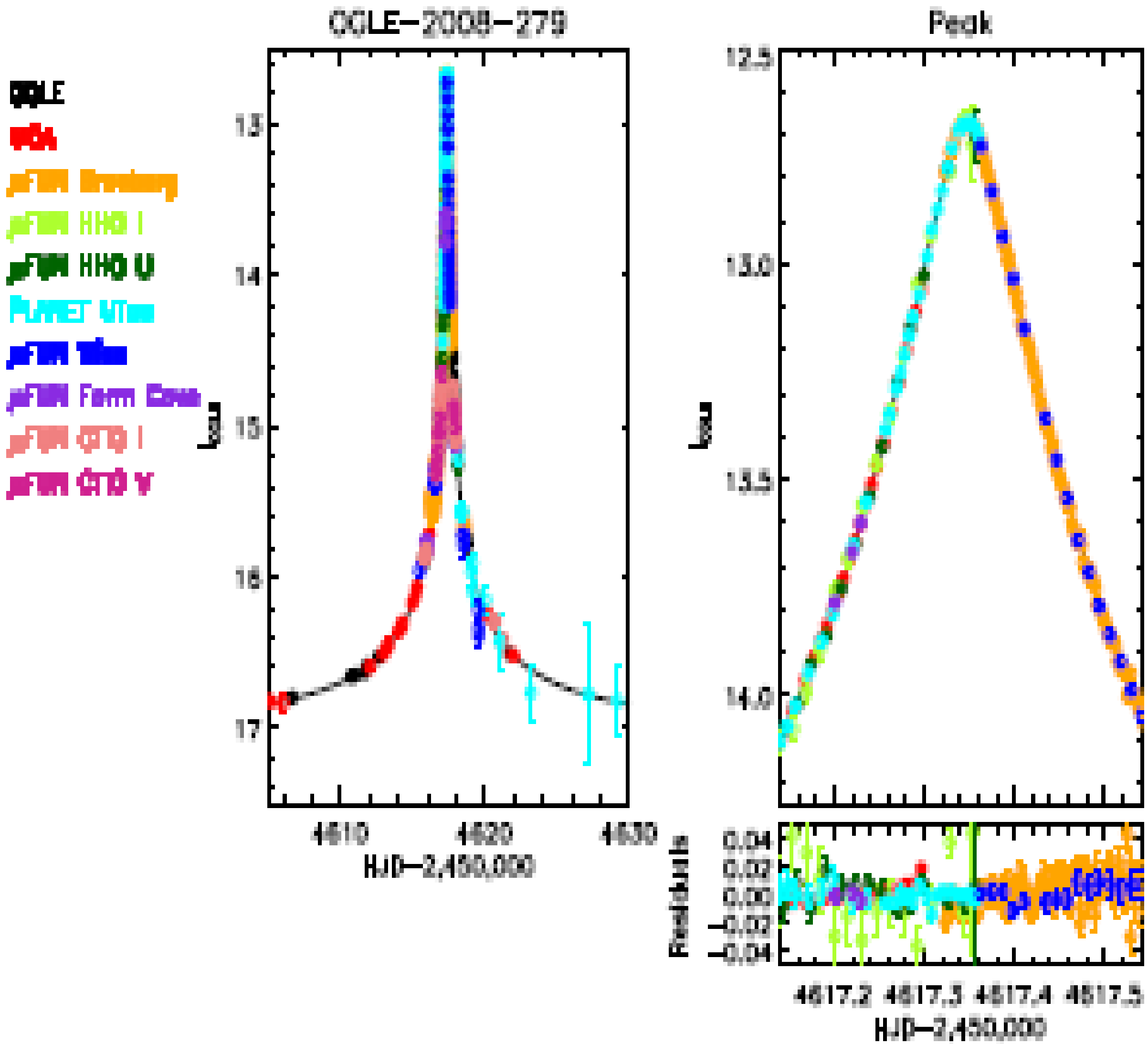
Canaries South Africa Chile



Terrestrial Parallax: Simultaneous Observations on Earth



OGLE-2008-BLG-279:
A = 1600



Yee et al. 2009, ApJ, 730, 2082

Events with Terrestrial Parallax

Name	M	D_L	μ	θ_*	$\rho\tilde{r}_{\rm E}$	$t_{\rm E}$	$A_{\rm max}$
	($M_\odot$)	(kpc)	(mas yr^{-1})	(μas)	($R_\oplus$)	(day)	
OGLE-2007-BLG-224	0.056	0.5	48.0	0.77	10	106	2400
OGLE-2008-BLG-279	0.64	4.0	2.7	0.54	100	7	1600

Gould & Yee 2012 ApJ, 764, 107

Events with Terrestrial Parallax

Name	M	D_L	μ	θ_*	$\rho\tilde{r}_E$	t_E	A_{max}
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$$\pi_{\rm rel} = \theta_E \pi_E = \frac{\rm AU}{\rho \tilde{r}_E}\theta_* \gtrsim 0.28\,{\rm mas}\frac{\theta_*}{0.6\,\mu{\rm as}}$$

Gould & Yee 2012 ApJ, 764, 107

Events with Terrestrial Parallax

Name	M	D_L	μ	θ_*	$\rho\tilde{r}_{\rm E}$	$t_{\rm E}$	$A_{\rm max}$
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$$\pi_{\rm rel} = \theta_{\rm E}\pi_{\rm E} = \frac{\rm AU}{\rho\tilde{r}_{\rm E}}\theta_* \gtrsim 0.28\,{\rm mas}\frac{\theta_*}{0.6\,\mu{\rm as}}$$

$$\Gamma = 2\langle\mu\rangle\theta_*\int_0^{D_{\rm max}} dD_L D_L^2 n(D_L) = 1.6\,{\rm Gyr}^{-1}$$
$$\times\left(\frac{\langle\mu\rangle}{10\,{\rm mas\,yr}^{-1}}\right)\left(\frac{\theta_*}{0.6\,\mu{\rm as}}\right)\left(\frac{D_{\rm max}}{2.5\,{\rm kpc}}\right)^3\left(\frac{\langle n\rangle}{1\,{\rm pc}^{-3}}\right)$$

Gould & Yee 2012 ApJ, 764, 107

Events with Terrestrial Parallax

Name	M ($M_\odot$)	D_L (kpc)	μ (mas yr^{-1})	θ_* (μas)	$\rho\tilde{r}_E$ ($R_\oplus$)	t_E (day)	A_{max}
OGLE-2007-BLG-224	0.056	0.5	48.0	0.77	10	106	2400
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$$\pi_{\rm rel} = \theta_{\rm E}\pi_{\rm E} = \frac{\rm AU}{\rho\tilde{r}_{\rm E}}\theta_* \gtrsim 0.28\,{\rm mas}\frac{\theta_*}{0.6\,\mu{\rm as}}$$

$$\Gamma = 2\langle\mu\rangle\theta_* \int_0^{D_{\rm max}} dD_L D_L^2 n(D_L) = 1.6\,{\rm Gyr}^{-1} \times \left(\frac{\langle\mu\rangle}{10\,{\rm mas\,yr}^{-1}}\right)\left(\frac{\theta_*}{0.6\,\mu{\rm as}}\right)\left(\frac{D_{\rm max}}{2.5\,{\rm kpc}}\right)^3\left(\frac{\langle n\rangle}{1\,{\rm pc}^{-3}}\right)$$

$$(1/4)(1/10)(1/2)\Gamma N T = 0.1$$

(N = 5e8; T = 10 yr)

Gould & Yee 2012 ApJ, 764, 107

Binary-Lens Equation

$$\mathbf{u} - \mathbf{y} = -\frac{\mathbf{y} - \mathbf{y}_L}{|\mathbf{y} - \mathbf{y}_L|^2}$$

$$\mathbf{y}_L = 0 \rightarrow \mathbf{u} - \mathbf{y} = -\frac{\mathbf{y}}{y^2} \Longrightarrow u - y = -\frac{1}{y}$$

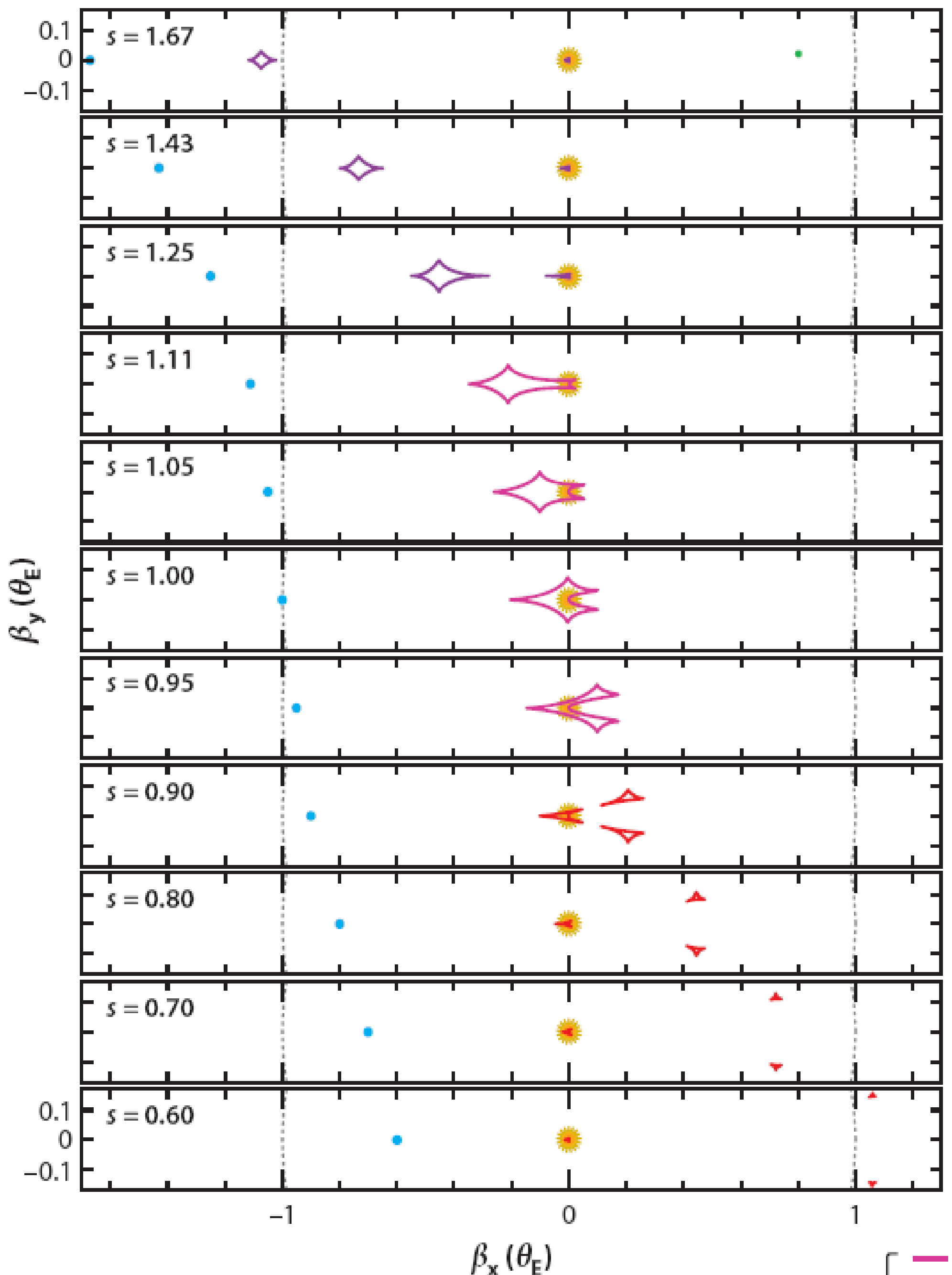
$$\Longrightarrow (y - u)y = 1 \Longrightarrow (\theta_I - \theta_S)\theta_I = \theta_{\mathrm{E}}^2$$

$$\mathbf{u} = \mathbf{y} - \sum_i \epsilon_i \frac{\mathbf{y} - \mathbf{y}_{m,i}}{|\mathbf{y} - \mathbf{y}_{m,i}|^2} \qquad \epsilon_i \equiv \frac{m_i}{M_{\mathrm{tot}}}$$

$$\zeta = z - \sum_i \frac{\epsilon_i}{\bar{z} - \bar{z}_{m,i}}$$

$$\zeta \equiv u_1 + iu_2 \qquad z \equiv y_1 + iy_2$$

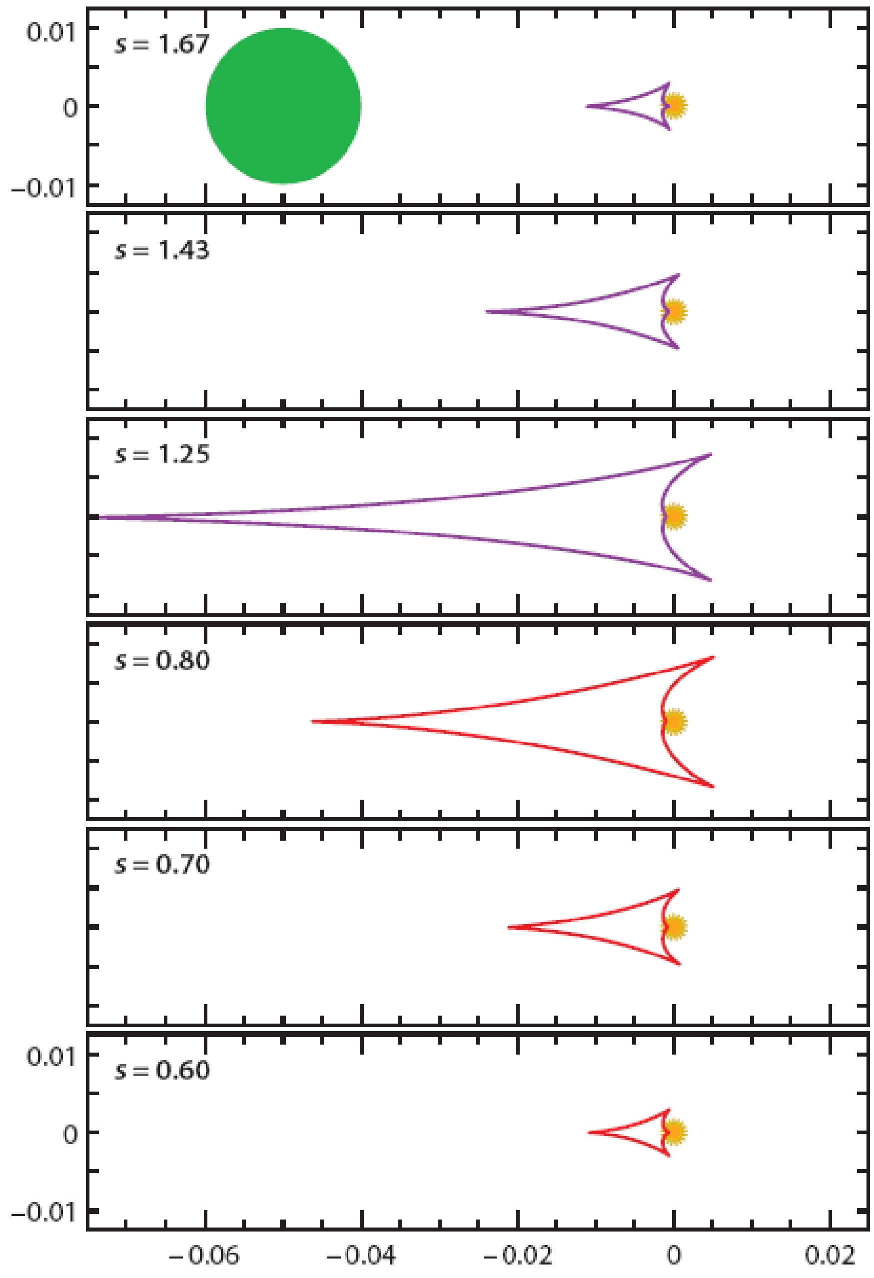
Binary-Lens Topologies



6 Features & 6 Parameters

6 Features	6 Parameters
Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q

Planetary Central Caustics



7 Features & 7 Parameters

Time of Peak	t_0
Height of Peak	u_0
Width of Peak	t_E
Time of Perturbation	Trajectory angle: α
Height of Perturbation	Planet-star separation: s
Width of Perturbation	Planet/star mass ratio: q
Smearing of Caustic	$t_* = \rho * t_E$