Problems Day 2
PhD school: Vietri Sul Mare 2018

Problem 1: Majoranas and quantum spin Hall edges

Let consider a quantum spin Hall edge with helical edge modes in a magnetic field (aligned perpendicular to the spin orbit field) and proximity coupled to a superconducor. Its Bogoliubov-deGennes Hamiltonian takes the form

$$\mathcal{H} = [up\sigma_x - \mu]\,\tau_z - B\sigma_z + \Delta\tau_x, \tag{1}$$

where u is the edge mode velocity.

(a) Compute the spectrum for small p. A convenient trick uses the fact that the spectrum is symmetric about $E = 0$. Thus, we can square the Hamiltonian and study its eigenvalues. Show that

$$\mathcal{H}^2 = (up)^2 + B^2 + \Delta^2 + \mu^2 - 2\mu(up)\sigma_x - 2B\Delta\sigma_z\tau_x + 2B\mu\sigma_z\tau_z. \tag{2}$$

First, consider the case $\mu = 0$ and show that the spectrum is given by

$$E_p^2 = (up)^2 + (B \pm \Delta)^2. \tag{3}$$

Note that the gap closes for $B = \Delta$, signifying the topological phase transition.

(b) To find the spectrum for nonzero μ, evaluate

$$\{\mathcal{H}^2 - [(up)^2 + B^2 + \Delta^2 + \mu^2]\}^2 \tag{4}$$

and find

$$E_p^2 = (up)^2 + B^2 + \Delta^2 + \mu^2 \pm \sqrt{(2\mu up)^2 + (2B\Delta)^2 + (2B\mu)^2}. \tag{5}$$

Use this result to show that the gap is given by

$$\text{gap} = |B - \sqrt{\Delta^2 + \mu^2}|, \tag{6}$$

which shows that the topological phase transition can be induced by variation of the chemical potential.

(c) Our treatment of small p so far still involves both low energy (of order $|B - \Delta|$) and high energy (of order $|B+\Delta|$) excitations. We can also strictly project the Hamiltonian to low energies by expanding about the critical point $B = \Delta$. To this end, we write the Hamiltonian as

$$\mathcal{H} = \mathcal{H}_0 + b\sigma_z - \mu\tau_z \tag{7}$$

with $b = B - \Delta$ and the Hamiltonian

$$\mathcal{H}_0 = up\sigma_x\tau_z - \Delta\sigma_z + \Delta\tau_x \tag{8}$$

exactly at the topological critical point. First show that the eigenspinors of $\mathcal{H}_0$, coresponding to two counterpropagating gapless low-energy modes with energies $E_{p,\pm} = \pm up$ are given by

$$|p,+\rangle = \frac{1}{2}[1,1,1,-1]^T \qquad |p,-\rangle = \frac{1}{2}[1,-1,1,1]^T. \tag{9}$$

This can be seen most easily by inserting these solutions into the eigenvalue equations. Note that these eigenspinors correspond to Majorana spinors! Thus, the theory reduces to two counterpropagating Majorana modes in the vicinity of the topological phase transition.

Now consider the matrix elements $\langle p, +|\mathcal{H}|p, +\rangle$, $\langle p, -|\mathcal{H}|p, -\rangle$, and $\langle p, +|\mathcal{H}|p, -\rangle$ to show that in the low energy subspace, the low energy Hamiltonian takes the form

$$\mathcal{H} \simeq \begin{pmatrix} up & -b \\ -b & -up \end{pmatrix}. \tag{10}$$

Note that a nonzero, but small μ does not enter into the low energy Hamiltonian in linear order. Compare this result to the low energy Hamiltonina in problem 3 of set 1.

Further optional problems: In these problems, you explore the model for semiconductor quantum wires proximity coupled to an s-wave superconductor in more detail by discussing the corresponding Bogoliubov-deGennes Hamiltonian

$$\mathcal{H} = \left[\frac{p^2}{2m} + up\sigma_x - \mu\right]\tau_z - B\sigma_z + \Delta\tau_x \tag{11}$$

Here, u denotes the strength of the spin-orbit coupling, σ_i and τ_i denote Pauli matrices in spin and particle-hole space, μ is the chemical potential, and Δ the strength of the proximity-induced superconductivity. This way of writing the Bogoliubov-deGennes Hamiltonian assumes that the Nambu spinor is taken in the form $\Psi = [\psi_\uparrow, \psi_\downarrow, \psi_\downarrow^\dagger, -\psi_\uparrow^\dagger]^T$.

In principle, it is a trivial matter to diagonalize this 4×4 Hamiltonian directly and if you are interested, I would encourage you to do that. However, this should reasonably be done on a computer. Here, I want to discuss limiting cases which can be analyzed analytically, which are instructive for thinking about the problem, and which are helpful for solving more advanced problems such as the effects of disorder and interactions.

Problem 2: Normal Hamiltonian of semiconductor wires proximity coupled to s-wave superconductors

Consider the system in the absence of the proximity coupling so that the electron Hamiltonian takes the form

$$H_0 = \frac{p^2}{2m} + up\sigma_x - B\sigma_z. \tag{12}$$

(a) Compute and sketch the spectrum for $u = B = 0$, for $B = 0$, and for the full Hamiltonian H_0. In the last case, make sure to plot the two regimes $B \ll \epsilon_{SO} = mu^2$ and $B \gg \epsilon_{SO}$ separately. Also discuss the spin orientation of the electrons at the Fermi energy in these two regimes.

(b) In order to find the eigenspinors of the Hamiltonian, consider a rotation in spin space about the y-axis, implemented by the unitary transformation

$$U = \exp(-i\alpha\sigma_y/2) = \cos\frac{\alpha}{2} - i\sigma_y \sin\frac{\alpha}{2}. \tag{13}$$

Show that this unitary transformation transforms the Hamiltonian into diagonal form,

$$H_0 \to UH_0U^\dagger = \frac{p^2}{2m} - \sqrt{(up)^2 + B^2}\,\sigma_z, \tag{14}$$

when choosing $\tan\alpha = up/B$ (and thus $\sin\alpha = up/\sqrt{(up)^2 + B^2}$ and $\cos\alpha = B/\sqrt{(up)^2 + B^2}$). Use this result to show that the eigenspinors of H_0 are given by

$$|\uparrow\rangle_e = U^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) \\ -\sin(\alpha/2) \end{pmatrix} \tag{15}$$

$$|\downarrow\rangle_e = U^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin(\alpha/2) \\ \cos(\alpha/2) \end{pmatrix} \tag{16}$$

with eigenenergies $\frac{p^2}{2m} \mp \sqrt{(up)^2 + B^2}$.

(c) Show that the corresponding hole spinors (i.e., the eigenspinors of the Hamiltonian $H_0 = -\frac{p^2}{2m} - up\sigma_x - B\sigma_z$) follow from the electron eigenspinors by taking $p \to -p$ and thus from $\alpha \to -\alpha$ so that

$$| \uparrow\rangle_h = \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{pmatrix} \tag{17}$$

$$| \downarrow\rangle_h = \begin{pmatrix} -\sin(\alpha/2) \\ \cos(\alpha/2) \end{pmatrix} \tag{18}$$

with eigenenergies $-[\frac{p^2}{2m} \pm \sqrt{(up)^2 + B^2}]$. We will use these eigenspinors for electrons and holes in the next problem.

Problem 3: Mapping to spinless p-wave superconductor

First discuss the situation $B \gg \Delta$ and let's choose $\mu = 0$ for simplicity. In this limit, we can map the Hamiltonian of the wire to the spinless p-wave superconductor discussed in the first problem set. The basic idea is that the normal Hamiltonian has two bands and we can project the Bogoliubov-deGennes Hamiltonian onto the lower band since the upper band is far above the Fermi energy for all momenta. The low-energy spinors are $| \uparrow\rangle_e$ for electrons and $| \downarrow\rangle_h$ for holes, which we derived explicitly in the previous problem and which for brevity we will refer to as $|e\rangle$ and $|h\rangle$ in the following.

(a) Compute the matrix elements $\langle e|\mathcal{H}|e\rangle$, $\langle e|\mathcal{H}|h\rangle$, and $\langle h|\mathcal{H}|h\rangle$ of the Bogoliubov-deGennes Hamiltonian $\mathcal{H}$ to show that the projected Hamiltonian takes the form

$$\mathcal{H} \simeq \left(\frac{p^2}{2m} - \sqrt{(up)^2 + B^2}\right)\tau_z - \frac{up}{\sqrt{(up)^2 + B^2}}\Delta\tau_x. \tag{19}$$

(b) Show that this reduces to

$$\mathcal{H} \simeq \left(\frac{p^2}{2m} - B\right)\tau_z - \frac{up}{B}\Delta\tau_x. \tag{20}$$

for $B \gg \epsilon_{SO}$ and to

$$\mathcal{H} \simeq \left(\frac{p^2}{2m} - u|p|\right)\tau_z - \mathrm{sgn}p\Delta\tau_x. \tag{21}$$

for $B \ll \epsilon_{SO}$. Explain physically why the magnitude of the gap is different in the two limits. You should find it useful to remember the discussion of the spin orientations in the previous problem.